

Prolate spheroidal wave functions (PSWFs) play an important role in various areas, from physics (e.g. wave phenomena, fluid dynamics) to engineering (e.g. signal processing, filter design). Even though the significance of PSWFs was realized at least half a century ago, and they frequently occur in applications, their analytical properties have not been investigated as much as those of many other special functions. In particular, despite some recent progress, the gap between asymptotic expansions and numerical experience, on the one hand, and rigorously proven explicit bounds and estimates, on the other hand, is still rather wide.

This paper attempts to improve the current situation. We analyze the differential operator associated with PSWFs, to derive fairly tight estimates on its eigenvalues. By combining these inequalities with a number of standard techniques, we also obtain several other properties of the PSWFs. The results are illustrated via numerical experiments.

## Non-asymptotic Analysis of Bandlimited Functions

Andrei Osipov<sup>†</sup>

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## 1 Introduction

The principal goal of this paper is non-asymptotic analysis of bandlimited functions. A function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is bandlimited of band limit  $c > 0$ , if there exists a function  $\sigma \in L^2[-1, 1]$  such that

$$f(x) = \int_{-1}^1 \sigma(t) e^{icxt} dt. \quad (1)$$

In other words, the Fourier transform of a bandlimited function is compactly supported. While (1) defines  $f$  for all real  $x$ , one is often interested in bandlimited functions, whose argument is confined to an interval, e.g.  $-1 \leq x \leq 1$ . Such functions are encountered in physics (wave phenomena, fluid dynamics), engineering (signal processing), etc. (see e.g. [14], [19], [20]).

About 50 years ago it was observed that the eigenfunctions of the integral operator  $F_c : L^2[-1, 1] \rightarrow L^2[-1, 1]$ , defined via the formula

$$F_c[\varphi](x) = \int_{-1}^1 \varphi(t) e^{icxt} dt, \quad (2)$$

provide a natural tool for dealing with bandlimited functions, defined on the interval  $[-1, 1]$ . Moreover, it was observed (see [9], [10], [12]) that the eigenfunctions of  $F_c$  are precisely the prolate spheroidal wave functions (PSWFs), well known from the mathematical physics [16], [19]. The PSWFs are the eigenfunctions of the differential operator  $L_c$ , defined via the formula

$$L_c[\varphi](x) = -\frac{d}{dx} \left( (1-x^2) \cdot \frac{d\varphi}{dx}(x) \right) + c^2 x^2. \quad (3)$$

In other words, the integral operator  $F_c$  commutes with the differential operator  $L_c$  [9], [18]. This property, being remarkable by itself, also plays an important role in both the analysis of PSWFs and the associated numerical algorithms [3], [4].

It is perhaps surprising, however, that the analytical properties of PSWFs have not been investigated as thoroughly as those of several other classes of special functions. In particular, when one reads through the classical works about the PSWFs [9], [10], [11], [12], [13], one is amazed by the number of properties stated without rigorous proofs. Some other properties are only supported by analysis of an asymptotic nature; see, for example, [6], [7], [15], [17]. This problem has been addressed in a number of recently published papers, for example, [2], [4], [5]. Still, the gap between numerical experience and asymptotic expansions, on the one hand, and rigorously proven explicit bounds and estimates, on the other hand, is rather wide; this paper offers a partial remedy for this deficiency.

This paper is mostly devoted to the analysis of the differential operator  $L_c$ , defined via (3). In particular, several explicit bounds for the eigenvalues of  $L_c$  are derived. These bounds turn out to be fairly tight, and the resulting inequalities lead to rigorous proofs of several other properties of PSWFs. The analysis is supported by and is illustrated through several numerical experiments.

The analysis of the eigenvalues of the integral operator  $F_c$ , defined via (2), requires tools different from those used in this paper; it will be published at a later date. The implications of the analysis of both  $L_c$  and  $F_c$  to numerical algorithms involving PSWFs are being currently investigated.

This paper is organized as follows. In Section 2, we summarize a number of well known mathematical facts to be used in the rest of this paper. In Section 3, we provide a summary of the principal results of this paper. In Section 4, we introduce the necessary analytical apparatus and carry out the analysis. In Section 5, we illustrate the analysis via several numerical examples.

## 2 Mathematical and Numerical Preliminaries

In this section, we introduce notation and summarize several facts to be used in the rest of the paper.

### 2.1 Prolate Spheroidal Wave Functions

In this subsection, we summarize several facts about the PSWFs. Unless stated otherwise, all these facts can be found in [4], [5], [7], [9], [10].

Given a real number  $c > 0$ , we define the operator  $F_c : L^2[-1, 1] \rightarrow L^2[-1, 1]$  via the formula

$$F_c[\varphi](x) = \int_{-1}^1 \varphi(t) e^{icxt} dt. \quad (4)$$

Obviously,  $F_c$  is compact. We denote its eigenvalues by  $\lambda_0, \lambda_1, \dots, \lambda_n, \dots$  and assume that they are ordered such that  $|\lambda_n| \geq |\lambda_{n+1}|$  for all natural  $n \geq 0$ . We denote by  $\psi_n$  the eigenfunction corresponding to  $\lambda_n$ . In other words, the following identity holds for all

integer  $n \geq 0$  and all real  $-1 \leq x \leq 1$ :

$$\lambda_n \psi_n(x) = \int_{-1}^1 \psi_n(t) e^{icxt} dt. \quad (5)$$

We adopt the convention<sup>1</sup> that  $\|\psi_n\|_{L^2[-1,1]} = 1$ . The following theorem describes the eigenvalues and eigenfunctions of  $F_c$ .

**Theorem 1.** *Suppose that  $c > 0$  is a real number, and that the operator  $F_c$  is defined via (4) above. Then, the eigenfunctions  $\psi_0, \psi_1, \dots$  of  $F_c$  are purely real, are orthonormal and are complete in  $L^2[-1, 1]$ . The even-numbered functions are even, the odd-numbered ones are odd. Each function  $\psi_n$  has exactly  $n$  simple roots in  $(-1, 1)$ . All eigenvalues  $\lambda_n$  of  $F_c$  are non-zero and simple; the even-numbered ones are purely real and the odd-numbered ones are purely imaginary; in particular,  $\lambda_n = i^n |\lambda_n|$ .*

We define the self-adjoint operator  $Q_c : L^2[-1, 1] \rightarrow L^2[-1, 1]$  via the formula

$$Q_c[\varphi](x) = \frac{1}{\pi} \int_{-1}^1 \frac{\sin(c(x-t))}{x-t} \varphi(t) dt. \quad (6)$$

Clearly, if we denote by  $\mathcal{F} : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  the unitary Fourier transform, then

$$Q_c[\varphi](x) = \chi_{[-1,1]}(x) \cdot \mathcal{F}^{-1}[\chi_{[-c,c]}(\xi) \cdot \mathcal{F}[\varphi](\xi)](x), \quad (7)$$

i.e.  $Q_c$  represents low-passing followed by time-limiting.  $Q_c$  relates to  $F_c$ , defined via (4), by

$$Q_c = \frac{c}{2\pi} \cdot F_c^* \cdot F_c, \quad (8)$$

and the eigenvalues  $\mu_n$  of  $Q_n$  satisfy the identity

$$\mu_n = \frac{c}{2\pi} \cdot |\lambda_n|^2, \quad (9)$$

for all integer  $n \geq 0$ . Moreover,  $Q_c$  has the same eigenfunctions  $\psi_n$  as  $F_c$ . In other words,

$$\mu_n \psi_n(x) = \frac{1}{\pi} \int_{-1}^1 \frac{\sin(c(x-t))}{x-t} \psi_n(t) dt, \quad (10)$$

for all integer  $n \geq 0$  and all  $-1 \leq x \leq 1$ . Also,  $Q_c$  is closely related to the operator  $P_c : L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$ , defined via the formula

$$P_c[\varphi](x) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin(c(x-t))}{x-t} \varphi(t) dt, \quad (11)$$

which is a widely known orthogonal projection onto the space of functions of band limit  $c > 0$  on the real line  $\mathbb{R}$ .

The following theorem about the eigenvalues  $\mu_n$  of the operator  $Q_c$ , defined via (6), can be traced back to [7]:

---

<sup>1</sup> This convention agrees with that of [4], [5] and differs from that of [9].

**Theorem 2.** Suppose that  $c > 0$  and  $0 < \alpha < 1$  are positive real numbers, and that the operator  $Q_c : L^2[-1, 1] \rightarrow L^2[-1, 1]$  is defined via (6) above. Suppose also that the integer  $N(c, \alpha)$  is the number of the eigenvalues  $\mu_n$  of  $Q_c$  that are greater than  $\alpha$ . In other words,

$$N(c, \alpha) = \max \{k = 1, 2, \dots : \mu_{k-1} > \alpha\}. \quad (12)$$

Then,

$$N(c, \alpha) = \frac{2}{\pi}c + \left( \frac{1}{\pi^2} \log \frac{1-\alpha}{\alpha} \right) \log c + O(\log c). \quad (13)$$

According to (13), there are about  $2c/\pi$  eigenvalues whose absolute value is close to one, order of  $\log c$  eigenvalues that decay exponentially, and the rest of them are very close to zero.

The eigenfunctions  $\psi_n$  of  $Q_c$  turn out to be the PSWFs, well known from classical mathematical physics [16]. The following theorem, proved in a more general form in [12], formalizes this statement.

**Theorem 3.** For any  $c > 0$ , there exists a strictly increasing unbounded sequence of positive numbers  $\chi_0 < \chi_1 < \dots$  such that, for each integer  $n \geq 0$ , the differential equation

$$(1 - x^2) \psi''(x) - 2x \cdot \psi'(x) + (\chi_n - c^2 x^2) \psi(x) = 0 \quad (14)$$

has a solution that is continuous on  $[-1, 1]$ . Moreover, all such solutions are constant multiples of the eigenfunction  $\psi_n$  of  $F_c$ , defined via (4) above.

For all real  $c > 0$  and all integer  $n \geq 0$ , the following inequality holds:

$$n(n+1) < \chi_n < n(n+1) + c^2. \quad (15)$$

The following result provides an upper bound on  $\psi_n^2(1)$ .

**Theorem 4.** For all  $c > 0$  and all natural  $n \geq 0$ ,

$$\psi_n^2(1) < n + \frac{1}{2}. \quad (16)$$

## 2.2 Elliptic Integrals

In this subsection, we summarize several facts about elliptic integrals. These facts can be found, for example, in section 8.1 in [8], and in [21].

The incomplete elliptic integrals of the first and second kind are given, respectively, by the formulae

$$F(y, k) = \int_0^y \frac{dt}{\sqrt{1 - k^2 \sin^2 t}}, \quad (17)$$

$$E(y, k) = \int_0^y \sqrt{1 - k^2 \sin^2 t} dt, \quad (18)$$

where  $0 \leq y \leq \pi/2$  and  $0 \leq k \leq 1$ . By performing the substitution  $x = \sin t$ , we can write (17) and (18) as

$$F(y, k) = \int_0^{\sin(y)} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}, \quad (19)$$

$$E(y, k) = \int_0^{\sin(y)} \sqrt{\frac{1-k^2x^2}{1-x^2}} dx. \quad (20)$$

The complete elliptic integrals of the first and second kind are given, respectively, by the formulae

$$F(k) = F\left(\frac{\pi}{2}, k\right) = \int_0^{\pi/2} \frac{dt}{\sqrt{1-k^2 \sin^2 t}}, \quad (21)$$

$$E(k) = E\left(\frac{\pi}{2}, k\right) = \int_0^{\pi/2} \sqrt{1-k^2 \sin^2 t} dt, \quad (22)$$

where  $0 \leq k \leq 1$ .

### 2.3 Oscillation Properties of Second Order ODEs

In this subsection, we state several well known facts from the general theory of second order ordinary differential equations (see e.g. [1]).

The following two theorems appear in Section 3.6 of [1] in a slightly different form.

**Theorem 5** (distance between roots). *Suppose that  $h(t)$  is a solution of the ODE*

$$y''(t) + Q(t)y(t) = 0. \quad (23)$$

*Suppose also that  $x < y$  are two consecutive roots of  $h(t)$ , and that*

$$A^2 \leq Q(t) \leq B^2, \quad (24)$$

*for all  $x \leq t \leq y$ . Then,*

$$\frac{\pi}{B} < y - x < \frac{\pi}{A}. \quad (25)$$

**Theorem 6.** *Suppose that  $a < b$  are real numbers, and that  $g : (a, b) \rightarrow \mathbb{R}$  is a continuous monotone function. Suppose also that  $y(t)$  is a solution of the ODE*

$$y''(t) + g(t) \cdot y(t) = 0, \quad (26)$$

*in the interval  $(a, b)$ . Suppose furthermore that*

$$t_1 < t_2 < t_3 < \dots \quad (27)$$

are consecutive roots of  $y(t)$ . If  $g$  is non-decreasing, then

$$t_2 - t_1 \geq t_3 - t_2 \geq t_4 - t_3 \geq \dots \quad (28)$$

If  $g$  is non-increasing, then

$$t_2 - t_1 \leq t_3 - t_2 \leq t_4 - t_3 \leq \dots \quad (29)$$

The following theorem is a special case of Theorem 6.2 from Section 3.6 in [1]:

**Theorem 7.** *Suppose that  $g_1, g_2$  are continuous functions, and that, for all real  $t$  in the interval  $(a, b)$ , the inequality  $g_1(t) < g_2(t)$  holds. Suppose also that the function  $\phi_1, \phi_2$  satisfy, for all  $a < t < b$ ,*

$$\begin{aligned} \phi_1''(t) + g_1(t) \cdot \phi_1(t) &= 0, \\ \phi_2''(t) + g_2(t) \cdot \phi_2(t) &= 0. \end{aligned} \quad (30)$$

*Then,  $\phi_2$  has a root between every two consecutive roots of  $\phi_1$ .*

**Corollary 1.** *Suppose that the functions  $\phi_1, \phi_2$  are those of Theorem 7 above. Suppose also that*

$$\phi_1(t_0) = \phi_2(t_0), \quad \phi_1'(t_0) = \phi_2'(t_0), \quad (31)$$

*for some  $a < t_0 < b$ . Then,  $\phi_2$  has at least as many roots in  $(t_0, b)$  as  $\phi_1$ .*

*Proof.* By Theorem 7, we only need to show that if  $t_1$  is the minimal root of  $\phi_1$  in  $(t_0, b)$ , then there exists a root of  $\phi_2$  in  $(t_0, t_1)$ . By contradiction, suppose that this is not the case. In addition, without loss of generality, suppose that  $\phi_1(t), \phi_2(t)$  are positive in  $(t_0, t_1)$ . Then, due to (30),

$$\phi_1''\phi_2 - \phi_2''\phi_1 = (g_2 - g_1)\phi_1\phi_2, \quad (32)$$

and hence

$$\begin{aligned} 0 &< \int_{t_0}^{t_1} (g_2(s) - g_1(s)) \phi_1(s) \phi_2(s) ds \\ &= [\phi_1'(s) \phi_2(s) - \phi_1(s) \phi_2'(s)]_{t_0}^{t_1} \\ &= \phi_1'(t_1) \phi_2(t_1) \leq 0, \end{aligned} \quad (33)$$

which is a contradiction. ■



## 2.4 Prüfer Transformations

In this subsection, we describe the classical Prüfer transformation of a second order ODE (see e.g. [1],[22]). Also, we describe a modification of Prüfer transformation, introduced in [3] and used in the rest of the paper.

Suppose that we are given the second order ODE

$$\frac{d}{dt} (p(t)u'(t)) + q(t)u(t) = 0, \quad (34)$$

where  $t$  varies over some interval  $I$  in which  $p$  and  $q$  are continuously differentiable and have no roots. We define the function  $\theta : I \rightarrow \mathbb{R}$  via

$$\frac{p(t)u'(t)}{u(t)} = \gamma(t) \tan \theta(t), \quad (35)$$

where  $\gamma : I \rightarrow \mathbb{R}$  is an arbitrary positive continuously differentiable function. The function  $\theta(t)$  satisfies, for all  $t$  in  $I$ ,

$$\theta'(t) = -\frac{\gamma(t)}{p(t)} \sin^2 \theta(t) - \frac{q(t)}{\gamma(t)} \cos^2 \theta(t) - \left( \frac{\gamma'(t)}{\gamma(t)} \right) \frac{\sin(2\theta(t))}{2}. \quad (36)$$

One can observe that if  $u'(\tilde{t}) = 0$  for  $\tilde{t} \in I$ , then by (35)

$$\theta(\tilde{t}) = k\pi, \quad k \text{ is integer.} \quad (37)$$

Similarly, if  $u(\tilde{t}) = 0$  for  $\tilde{t} \in I$ , then

$$\theta(\tilde{t}) = (k + 1/2)\pi, \quad k \text{ is integer.} \quad (38)$$

The choice  $\gamma(t) = 1$  in (35) gives rise to the classical Prüfer transformation (see e.g. section 4.2 in [1]).

In [3], the choice  $\gamma(t) = \sqrt{q(t)p(t)}$  is suggested and shown to be more convenient numerically in several applications. In this paper, this choice also leads to a more convenient analytical tool than the classical Prüfer transformation.

Writing (14) in the form of (34) yields

$$p(t) = 1 - t^2, \quad q(t) = \chi_n - c^2 t^2, \quad (39)$$

for  $|t| < \min \{ \sqrt{\chi_n}/c, 1 \}$ . The equation (35) admits the form

$$\frac{p(t)\psi'_n(t)}{\psi_n(t)} = \sqrt{p(t)q(t)} \tan \theta(t), \quad (40)$$

which implies that

$$\theta(t) = \text{atan} \left( \sqrt{\frac{p(t)}{q(t)}} \frac{\psi'_n(t)}{\psi_n(t)} \right) + \pi m(t), \quad (41)$$

where  $m(t)$  is an integer determined for all  $t$  by an arbitrary choice at some  $t = t_0$  (the role of  $\pi m(t)$  in (41) is to enforce the continuity of  $\theta$  at the roots of  $\psi_n$ ). The first order ODE (36) admits the form (see [3], [22])

$$\theta'(t) = -f(t) + \sin(2\theta(t))v(t), \quad (42)$$

where the functions  $f, v$  are defined, respectively, via the formulae

$$f(t) = \sqrt{\frac{q(t)}{p(t)}} = \sqrt{\frac{\chi_n - c^2 t^2}{1 - t^2}} \quad (43)$$

and

$$v(t) = -\frac{1}{4} \cdot \frac{p(t)q'(t) + q(t)p'(t)}{p(t)q(t)} = \frac{1}{2} \left( \frac{t}{1 - t^2} + \frac{c^2 t}{\chi_n - c^2 t^2} \right). \quad (44)$$

### 3 Summary

In this section, we summarize some of the properties of prolate spheroidal wave functions (PSWFs), proved in Section 4. The PSWFs and the related notation were introduced in Section 2.1. Throughout this section, the band limit  $c > 0$  is assumed to be a fixed positive real number.

Many properties of the PSWF  $\psi_n$  depend on whether the eigenvalue  $\chi_n$  of the ODE (14) is greater than or less than  $c^2$ . The following simple relation between  $c, n$  and  $\chi_n$  is proved in Theorem 14 in Section 4.1.2.

**Proposition 1.** *Suppose that  $n \geq 2$  is a non-negative integer.*

- *If  $n \leq (2c/\pi) - 1$ , then  $\chi_n < c^2$ .*
- *If  $n \geq (2c/\pi)$ , then  $\chi_n > c^2$ .*
- *If  $(2c/\pi) - 1 < n < (2c/\pi)$ , then either inequality is possible.*

In the following proposition, we describe the location of “special points” (roots of  $\psi_n$ , roots of  $\psi'_n$ , turning points of the ODE (14)) that depends on whether  $\chi_n > c^2$  or  $\chi_n < c^2$ . It is proved in Lemma 1 in Section 4.1.1 and is illustrated on Figures 1, 2.

**Proposition 2.** *Suppose that  $n \geq 2$  is a positive integer. Suppose also that  $t_1 < \dots < t_n$  are the roots of  $\psi_n$  in  $(-1, 1)$ , and  $x_1 < \dots < x_{n-1}$  are the roots of  $\psi'_n$  in  $(t_1, t_n)$ . Suppose furthermore that the real number  $x_n$  is defined via the formula*

$$x_n = \begin{cases} \text{maximal root of } \psi'_n \text{ in } (-1, 1), & \text{if } \chi_n < c^2, \\ 1, & \text{if } \chi_n > c^2. \end{cases} \quad (45)$$

*Then,*

$$-\frac{\sqrt{\chi_n}}{c} < -x_n < t_1 < x_1 < t_2 < \dots < t_{n-1} < t_n < x_n < \frac{\sqrt{\chi_n}}{c}. \quad (46)$$

In particular, if  $\chi_n < c^2$ , then

$$t_n < x_n < \frac{\sqrt{\chi_n}}{c} < 1, \quad (47)$$

and  $\psi'_n$  has  $n + 1$  roots in the interval  $(-1, 1)$ ; and, if  $\chi_n > c^2$ , then

$$t_n < x_n = 1 < \frac{\sqrt{\chi_n}}{c}, \quad (48)$$

and  $\psi'_n$  has  $n - 1$  roots in the interval  $(-1, 1)$ .

The following two inequalities improve the inequality (15) in Section 2.1. Their proof can be found in Theorems 8,9 in Section 4.1.2. This is one of the principal analytical results of this paper. The inequalities (49), (50) below are illustrated in Tables 1, 2, 3, 4.

**Proposition 3.** Suppose that  $n \geq 2$  is a positive integer. Suppose also that  $t_n$  and  $T$  are the maximal roots of  $\psi_n$  and  $\psi'_n$  in the interval  $(-1, 1)$ , respectively. If  $\chi_n > c^2$ , then

$$1 + \frac{2}{\pi} \int_0^{t_n} \sqrt{\frac{\chi_n - c^2 t^2}{1 - t^2}} dt < n < \frac{2}{\pi} \int_0^1 \sqrt{\frac{\chi_n - c^2 t^2}{1 - t^2}} dt. \quad (49)$$

If  $\chi_n < c^2$ , then

$$1 + \frac{2}{\pi} \int_0^{t_n} \sqrt{\frac{\chi_n - c^2 t^2}{1 - t^2}} dt < n < \frac{2}{\pi} \int_0^T \sqrt{\frac{\chi_n - c^2 t^2}{1 - t^2}} dt. \quad (50)$$

Note that (49) and (50) differ only in the range of integration on their right-hand sides.

In the following proposition, we simplify the inequality (49) in Proposition 3. It is proven in Theorem 17 and Corollary 3 in Section 4.1.3.

**Proposition 4.** Suppose that  $n \geq 2$  is a positive integer, and that  $\chi_n > c^2$ . Then,

$$\begin{aligned} n &< \frac{2}{\pi} \int_0^1 \sqrt{\frac{\chi_n - c^2 t^2}{1 - t^2}} dt = \\ &\frac{2}{\pi} \sqrt{\chi_n} \cdot E\left(\frac{c}{\sqrt{\chi_n}}\right) < n + 3, \end{aligned} \quad (51)$$

where the function  $E : [0, 1] \rightarrow \mathbb{R}$  is defined via (22) in Section 2.2.

In the following proposition, we describe a relation between  $\chi_n$  and the maximal root  $t_n$  of  $\psi_n$  in  $(-1, 1)$ , by providing a lower and upper bounds on  $1 - t_n$  in terms of  $\chi_n$  and  $c$ . It is proved in Theorem 16, 18 in Section 4.1.3.

**Proposition 5.** Suppose that  $n \geq 2$  is a positive integer, and that  $\chi_n > c^2$ . Suppose also that  $t_n$  is the maximal root of  $\psi_n$  in the interval  $(-1, 1)$ . Then,

$$\begin{aligned} \frac{\pi^2/8}{\chi_n - c^2 + \sqrt{(\chi_n - c^2)^2 + (\pi c/2)^2}} &< 1 - t_n \\ &< \frac{4\pi^2}{\chi_n - c^2 + \sqrt{(\chi_n - c^2)^2 + (4\pi c)^2}}. \end{aligned} \quad (52)$$

In the following proposition, we provide yet another upper bound on  $\chi_n$  in terms of  $n$ . Its proof can be found in Theorem 12 in Section 4.1.3.

**Proposition 6.** *Suppose that  $n \geq 2$  is a positive integer, and that  $\chi_n > c^2$ . Then,*

$$\chi_n < \left( \frac{\pi}{2} (n+1) \right)^2. \quad (53)$$

We observe that, for sufficiently large  $n$ , the inequality (53) is even weaker than (15). On the other hand, (53) can be useful for  $n$  near  $2c/\pi$ , as illustrated in Tables 5, 6.

The following proposition summarizes Theorem 10 in Section 4.1.2 and Theorems 13, 15 in Section 4.1.3. It is illustrated in Tables 5, 6, 7, 8.

**Proposition 7.** *Suppose that  $n \geq 2$  is a positive integer, and that  $\chi_n > c^2$ . Suppose also that  $-1 < t_1 < t_2 < \dots < t_n < 1$  are the roots of  $\psi_n$  in the interval  $(-1, 1)$ . Suppose furthermore that the functions  $f, v$  are defined, respectively, via (43), (44) in Section 2.4. Then:*

- *For all integer  $(n+1)/2 \leq i \leq n-1$ , i.e. for all integer  $i$  such that  $0 \leq t_i < t_n$ ,*

$$\frac{\pi}{f(t_{i+1}) + v(t_{i+1})/2} < t_{i+1} - t_i < \frac{\pi}{f(t_i)}. \quad (54)$$

- *For all integer  $(n+1)/2 \leq i \leq n-1$ , i.e. for all integer  $i$  such that  $0 \leq t_i < t_n$ ,*

$$t_{i+1} - t_i > t_{i+2} - t_{i+1} > \dots > t_n - t_{n-1}. \quad (55)$$

- *For all integer  $j = 1, \dots, n-1$ ,*

$$t_{j+1} - t_j < \frac{\pi}{\sqrt{\chi_n + 1}}. \quad (56)$$

The following proposition summarizes Theorem 15 in Section 4.1.3.

**Proposition 8.** *Suppose that  $n \geq 2$  is an integer, and that  $\chi_n < c^2 - c\sqrt{2}$ . Suppose also that  $-1 < t_1 < t_2 < \dots < t_n < 1$  are the roots of  $\psi_n$  in the interval  $(-1, 1)$ . Then,*

$$t_{i+1} - t_i < t_{i+2} - t_{i+1} < \dots < t_n - t_{n-1}, \quad (57)$$

*for all integer  $(n+1)/2 \leq i \leq n-1$ , i.e. for all integer  $i$  such that  $0 \leq t_i < t_n$ .*

The following proposition summarizes Theorem 4 in Section 2.1 and Theorem 19 in Section 4.2.

**Proposition 9.** *Suppose that  $n \geq 0$  is a non-negative integer, and that  $\chi_n > c^2$ . Then,*

$$\frac{1}{2} < \psi_n^2(1) < n + \frac{1}{2}. \quad (58)$$

The following proposition is illustrated on Figures 1, 2. It is proved in Theorem 20 in Section 4.1.3.

**Proposition 10.** *Suppose that  $n \geq 0$  is a non-negative integer, and that  $x, y$  are two arbitrary extremum points of  $\psi_n$  in  $(-1, 1)$ . If  $|x| < |y|$ , then*

$$|\psi_n(x)| < |\psi_n(y)|. \quad (59)$$

*If, in addition,  $\chi_n > c^2$ , then*

$$|\psi_n(x)| < |\psi_n(y)| < |\psi_n(1)|. \quad (60)$$

## 4 Analytical Apparatus

The purpose of this section is to provide the analytical apparatus to be used in the rest of the paper, as well as to prove the results summarized in Section 3.

### 4.1 Oscillation Properties of PSWFs

In this subsection, we prove several facts about the distance between consecutive roots of PSWFs (5) and find a more subtle relationship between  $n$  and  $\chi_n$  (14) than the one given by (15). Throughout this subsection  $c > 0$  is a positive real number and  $n$  is a non-negative integer. The principal results of this subsection are Theorems 8, 9, 11, and 12.

#### 4.1.1 Special Points of $\psi_n$

We refer to the roots of  $\psi_n$ , the roots of  $\psi'_n$  and the turning points of the ODE (14) as "special points". Some of them play an important role in the subsequent analysis. These points are introduced in the following definition.

**Definition 1** (Special points). *Suppose that  $n \geq 2$  is a positive integer. We define*

- $t_1 < t_2 < \dots < t_n$  to be the roots of  $\psi_n$  in  $(-1, 1)$ ,
- $x_1 < \dots < x_{n-1}$  to be the roots of  $\psi'_n$  in  $(t_1, t_n)$ ,
- $x_n$  via the formula

$$x_n = \begin{cases} \text{maximal root of } \psi'_n \text{ in } (-1, 1), & \text{if } \chi_n < c^2, \\ 1, & \text{if } \chi_n > c^2. \end{cases} \quad (61)$$

This definition will be used throughout all of Section 4. The relative location of some of the special points depends on whether  $\chi_n > c^2$  or  $\chi_n < c^2$ . This is illustrated in Figures 1, 2 and is described by the following lemma.

**Lemma 1** (Special points). *Suppose that  $n \geq 2$  is a positive integer. Suppose also that  $t_1 < \cdots < t_n$  and  $x_1 < \cdots < x_n$  are those of Definition 1. Then,*

$$-\frac{\sqrt{\chi_n}}{c} < -x_n < t_1 < x_1 < t_2 < \cdots < t_{n-1} < t_n < x_n < \frac{\sqrt{\chi_n}}{c}. \quad (62)$$

*In particular, if  $\chi_n < c^2$ , then*

$$t_n < x_n < \frac{\sqrt{\chi_n}}{c} < 1, \quad (63)$$

*and  $\psi'_n$  has  $n+1$  roots in the interval  $(-1, 1)$ ; and, if  $\chi_n > c^2$ , then*

$$t_n < x_n = 1 < \frac{\sqrt{\chi_n}}{c}, \quad (64)$$

*and  $\psi'_n$  has  $n-1$  roots in the interval  $(-1, 1)$ .*

*Proof.* Without loss of generality, we assume that

$$\psi_n(1) > 0. \quad (65)$$

Obviously, (65) implies that

$$\psi'_n(t_n) > 0. \quad (66)$$

Suppose first that  $\chi_n < c^2$ . Then, due to the ODE (14) in Section 2.1,

$$\psi'_n(1) = \frac{\chi_n - c^2}{2} \cdot \psi_n(1) < 0. \quad (67)$$

We combine (14) and (66) to obtain

$$\psi''_n(t_n) = \frac{2t_n}{1 - t_n^2} \cdot \psi'_n(t_n) > 0. \quad (68)$$

In addition, we combine (61), (66), (67) to conclude that the maximal root  $x_n$  of  $\psi'_n$  in  $(-1, 1)$  satisfies

$$t_n < x_n < 1. \quad (69)$$

Moreover, (65) implies that, for any root  $x$  of  $\psi'_n$  in  $(t_n, 1)$ ,

$$\psi''_n(x) = -\frac{\chi_n - c^2 x^2}{1 - x^2} \cdot \psi_n(x) < 0. \quad (70)$$

We combine (65), (69), (70) with (14) to obtain

$$\frac{c^2 x_n^2 - \chi_n}{1 - x_n^2} = \frac{\psi''_n(x_n)}{\psi_n(x_n)} < 0, \quad (71)$$

which implies both (62) and (63). In addition, we combine (14), (63) and (70) to conclude that  $x_n$  is the only root of  $\psi'_n$  between  $t_n$  and 1. Thus,  $\psi'_n$  indeed has  $n+1$  roots in  $(-1, 1)$ .

Suppose now that  $\chi_n > c^2$ . We combine (14) and (65) to obtain

$$\psi'_n(1) = \frac{\chi_n - c^2}{2} \cdot \psi_n(1) > 0. \quad (72)$$

If  $t_n < x < 1$  is a root of  $\psi'_n$ , then

$$\psi''_n(x) = -\frac{\chi_n - c^2 x^2}{1 - x^2} \cdot \psi_n(x) < 0, \quad (73)$$

therefore  $\psi'_n$  can have at most one root in  $(t_n, 1)$ . We combine this observation with (61), (66), (72) and (73) to conclude that, in fact,  $\psi'_n$  has no roots in  $(t_n, 1)$ , and hence both (62) and (64) hold. In particular,  $\psi'_n$  has  $n-1$  roots in  $(-1, 1)$ .  $\blacksquare$

#### 4.1.2 A Sharper Inequality for $\chi_n$

In this subsection, we use the modified Prüfer transformation (see Section 2.4) to analyze the relationship between  $n, c$  and  $\chi_n$ . In particular, this analysis yields fairly tight lower and upper bounds on  $\chi_n$  in terms of  $c$  and  $n$ . These bounds are described in Theorems 8, 9 below. These theorems are not only one of the principal results of this paper, but are subsequently used in the proofs of Theorems 10, 11, 12, 17, 18.

We start with developing the required analytical machinery. In the following lemma, we describe several properties of the modified Prüfer transformation (see Section 2.4), applied to the prolate differential equation (14).

**Lemma 2.** *Suppose that  $n \geq 2$  is a positive integer. Suppose also that the numbers  $t_1, \dots, t_n$  and  $x_1, \dots, x_n$  are those of Definition 1 in Section 4.1.1, and that the function  $\theta : [-x_n, x_n] \rightarrow \mathbb{R}$  is defined via the formula*

$$\theta(t) = \begin{cases} (i - \frac{1}{2}) \cdot \pi, & \text{if } t = t_i \text{ for some } 1 \leq i \leq n, \\ \operatorname{atan}\left(-\sqrt{\frac{1-t^2}{\chi_n - c^2 t^2}} \cdot \frac{\psi'_n(t)}{\psi_n(t)}\right) + m(t) \cdot \pi, & \text{if } \psi_n(t) \neq 0, \end{cases} \quad (74)$$

where  $m(t)$  is the number of the roots of  $\psi_n$  in the interval  $(-1, t)$ . Then,  $\theta$  has the following properties:

- $\theta$  is continuously differentiable in the interval  $[-x_n, x_n]$ .
- $\theta$  satisfies, for all  $-x_n < t < x_n$ , the differential equation

$$\theta'(t) = f(t) + v(t) \cdot \sin(2\theta(t)), \quad (75)$$

where the functions  $f, v$  are defined, respectively, via (43), (44) in Section 2.4.

- for each integer  $0 \leq j \leq 2n$ , there is a unique solution to the equation

$$\theta(t) = j \cdot \frac{\pi}{2}, \quad (76)$$

for the unknown  $t$  in  $[-x_n, x_n]$ . More specifically,

$$\theta(-x_n) = 0, \quad (77)$$

$$\theta(t_i) = \left(i - \frac{1}{2}\right) \cdot \pi, \quad (78)$$

$$\theta(x_i) = i \cdot \pi, \quad (79)$$

for each  $i = 1, \dots, n$ . In particular,  $\theta(x_n) = n \cdot \pi$ .

*Proof.* We combine (62) in Lemma 1 with (74) to conclude that  $\theta$  is well defined for all  $-x_n \leq t \leq x_n$ , where  $x_n$  is given via (61) in Definition 1. Obviously,  $\theta$  is continuous, and the identities (77), (78), (79) follow immediately from the combination of Lemma 1 and (74). In addition,  $\theta$  satisfies the ODE (75) in  $(-x_n, x_n)$  due to (36), (40), (42) in Section 2.4.

Finally, to establish the uniqueness of the solution to the equation (76), we make the following observation. Due to (74), for any point  $t$  in  $(-x_n, x_n)$ , the value  $\theta(t)$  is an integer multiple of  $\pi/2$  if and only if  $t$  is either a root of  $\psi_n$  or a root of  $\psi'_n$ . We conclude the proof by combining this observation with (61), (77) and (79). ■

**Remark 1.** We observe that, due to (77), (78), (79), for all  $i = 1, \dots, n$ ,

$$\sin(2\theta(t_i)) = \sin(2\theta(x_i)) = 0, \quad (80)$$

where  $t_1, \dots, t_n, x_1, \dots, x_n$  are those of Definition 1 in Section 4.1.1, and  $\theta$  is that of Lemma 2. This observation will play an important role in the analysis of the ODE (75) throughout the rest of this section.

In the following lemma, we prove that  $\theta$  of Lemma 2 is monotonically increasing.

**Lemma 3.** Suppose that  $n \geq 2$  is a positive integer. Suppose also that the real number  $x_n$  and the function  $\theta : [-x_n, x_n] \rightarrow \mathbb{R}$  are those of Lemma 2 above. Then,  $\theta$  is strictly increasing in  $[-x_n, x_n]$ , in other words,

$$\theta'(t) > 0, \quad (81)$$

for all  $-x_n < t < x_n$ .

*Proof.* We first prove that

$$\frac{d}{dt} \left( \frac{v}{f} \right) (t) > 0, \quad (82)$$

for  $-x_n < t < x_n$ , where the functions  $f, v$  are defined, respectively, via (43), (44) in



Section 2.4. We differentiate  $v/f$  with respect to  $t$  to obtain

$$\begin{aligned}
\left(\frac{v}{f}\right)' &= -\left(\frac{p'q + q'p}{4pq} \cdot \sqrt{\frac{p}{q}}\right)' = -\left(\frac{p'q + q'p}{4q^{3/2}p^{1/2}}\right)' \\
&= \frac{q^{-3}p^{-1}}{4} \cdot \left[\left(\frac{3}{2}q^{1/2}p^{1/2}q' + \frac{1}{2}q^{3/2}p^{-1/2}p'\right)(p'q + q'p) - \right. \\
&\quad \left.(p''q + 2p'q' + q''p)q^{3/2}p^{1/2}\right] \\
&= \frac{q^{-5/2}p^{-3/2}}{4} \cdot \left[\left(\frac{3}{2}q'p + \frac{1}{2}p'q\right)(p'q + q'p) - pq(p''q + 2p'q' + q''p)\right] \\
&= \frac{q^{-5/2}p^{-3/2}}{4} \cdot \left[\frac{3}{2}p^2(q')^2 + \frac{1}{2}q^2(p')^2 - q^2pp'' - p^2qq''\right] > 0,
\end{aligned} \tag{83}$$

since, due to (39),

$$p(t) > 0, \quad p''(t) = -2 < 0, \quad q(t) > 0, \quad q''(t) = -2c^2 < 0. \tag{84}$$

We now proceed to prove (81) for  $0 < t < x_n$ . Suppose that, by contradiction, there exists  $0 < x < x_n$  such that

$$\theta'(x) < 0. \tag{85}$$

Combined with (75) in Lemma 2 above, (85) implies that

$$1 + \frac{v(x)}{f(x)} \cdot \sin(2\theta(x)) = \frac{f(x) + v(x) \cdot \sin(2\theta(x))}{f(x)} < 0, \tag{86}$$

and, in particular, that

$$\sin(2\theta(x)) < 0. \tag{87}$$

Due to Lemma 2 above, there exists an integer  $(n+1)/2 \leq i \leq n$  such that

$$\left(i - \frac{1}{2}\right) \cdot \pi < \theta(x) < i \cdot \pi. \tag{88}$$

Moreover, due to (77), (78), (79), (85), (87), (88), there exists a point  $y$  such that

$$0 \leq t_i < x < y < x_i \leq x_n, \tag{89}$$

and also

$$\theta(x) = \theta(y), \quad \theta'(y) > 0. \tag{90}$$

for otherwise (79) would be impossible. We combine (75) and (90) to obtain

$$1 + \frac{v(y)}{f(y)} \cdot \sin(2\theta(y)) = \frac{f(y) + v(y) \cdot \sin(2\theta(y))}{f(y)} = \frac{\theta'(y)}{f(y)} > 0, \tag{91}$$

in contradiction to (82), (86) and (87). This concludes the proof of (81) for  $0 < t < x_n$ . For  $-x_n < t < 0$ , the identity (81) follows now from the symmetry considerations.  $\blacksquare$

The right-hand side of the ODE (75) of Lemma 2 contains a monotone term and an oscillatory term. In the following lemma, we study the integrals of the oscillatory term between various special points, introduced in Definition 1 in Section 4.1.1.

**Lemma 4.** *Suppose that  $n \geq 2$  is an integer. Suppose also that the real numbers  $t_1 < \dots < t_n$  and  $x_1 < \dots < x_n$  are those of Definition 1 in Section 4.1.1, and the function  $\theta : [-x_n, x_n] \rightarrow \mathbb{R}$  is that of Lemma 2 above. Suppose furthermore that the function  $v$  is defined via (44) in Section 2.4. Then,*

$$\int_{x_i}^{t_{i+1}} v(t) \cdot \sin(2\theta(t)) dt > 0, \quad (92)$$

$$\int_{t_{i+1}}^{x_{i+1}} v(t) \cdot \sin(2\theta(t)) dt < 0, \quad (93)$$

$$\int_{x_i}^{x_{i+1}} v(t) \cdot \sin(2\theta(t)) dt < 0, \quad (94)$$

for all integer  $(n-1)/2 \leq i \leq n-1$ , i.e. for all integer  $i$  such that  $0 \leq x_i < x_n$ . Note that the integral in (94) is the sum of the integrals in (92) and (93).

*Proof.* Suppose that  $i$  is a positive integer such that  $(n-1)/2 \leq i \leq n-1$ . Suppose also that the function  $s : [0, n \cdot \pi] \rightarrow [-x_n, x_n]$  is the inverse of  $\theta$ . In other words, for all  $0 \leq \eta \leq n \cdot \pi$ ,

$$\theta(s(\eta)) = \eta. \quad (95)$$

Using (75), (78), (79) in Lemma 2, we expand the left-hand side of (92) to obtain

$$\begin{aligned} \int_{x_i}^{t_{i+1}} v(t) \cdot \sin(2\theta(t)) dt &= \\ \int_{\theta(x_i)}^{\theta(t_{i+1})} v(s(\eta)) \cdot \sin(2\eta) \cdot s'(\eta) d\eta &= \\ \int_{i \cdot \pi}^{(i+1/2) \cdot \pi} \frac{v(s(\eta)) \cdot \sin(2\eta) d\eta}{f(s(\eta)) + v(s(\eta)) \cdot \sin(2\eta)} &= \\ \int_0^{\pi/2} \frac{v(s(\eta + i \cdot \pi)) \cdot \sin(2\eta) d\eta}{f(s(\eta + i \cdot \pi)) + v(s(\eta + i \cdot \pi)) \cdot \sin(2\eta)}, \end{aligned} \quad (96)$$

from which (92) readily follows due to (44) in Section 2.4 and (81) in Lemma 3. By the same token, we expand the left-hand side of (93) to obtain

$$\begin{aligned} \int_{t_{i+1}}^{x_{i+1}} v(t) \cdot \sin(2\theta(t)) dt &= \\ \int_{(i+1/2) \cdot \pi}^{(i+1) \cdot \pi} \frac{v(s(\eta)) \cdot \sin(2\eta) d\eta}{f(s(\eta)) + v(s(\eta)) \cdot \sin(2\eta)} &= \\ - \int_0^{\pi/2} \frac{v(s(\eta + (i+1/2) \cdot \pi)) \cdot \sin(2\eta) d\eta}{f(s(\eta + (i+1/2) \cdot \pi)) - v(s(\eta + (i+1/2) \cdot \pi)) \cdot \sin(2\eta)}, \end{aligned} \quad (97)$$

which, combined with (44) in Section 2.4 and (81) in Lemma 3, implies (93). Finally, for all  $0 < \eta < \pi/2$ ,

$$\frac{\sin(2\eta)}{(f/v)(s(\eta + (i + 1/2) \cdot \pi)) - \sin(2\eta)} > \frac{\sin(2\eta)}{(f/v)(s(\eta + i \cdot \pi)) + \sin(2\eta)}, \quad (98)$$

since the function  $f/v$  is decreasing due to (82) in the proof of Lemma 3. The inequality (94) now follows from the combination of (96), (97) and (98).  $\blacksquare$

We are now ready to prove one of the principal results of this paper. It is illustrated in Tables 1, 2, 3, 4.

**Theorem 8.** *Suppose that  $n \geq 2$  is a positive integer. If  $\chi_n > c^2$ , then*

$$n < \frac{2}{\pi} \int_0^1 \sqrt{\frac{\chi_n - c^2 t^2}{1 - t^2}} dt. \quad (99)$$

*If  $\chi_n < c^2$ , then*

$$n < \frac{2}{\pi} \int_0^T \sqrt{\frac{\chi_n - c^2 t^2}{1 - t^2}} dt, \quad (100)$$

*where  $T$  is the maximal root of  $\psi'_n$  in  $(-1, 1)$ . Note that (99) and (100) differ only in the range of integration on their right-hand sides.*

*Proof.* Suppose that the real numbers

$$-1 \leq -x_n < t_1 < x_1 < t_2 < \cdots < t_{n-1} < x_{n-1} < t_n < x_n \leq 1 \quad (101)$$

are those of Definition 1 in Section 4.1.1, and the function  $\theta : [-x_n, x_n] \rightarrow \mathbb{R}$  is that of Lemma 2 above. Suppose also that the functions  $f, v$  are defined, respectively, via (43), (44) in Section 2.4. If  $n$  is even, then we combine (75), (78), (79) in Lemma 2 with (94) in Lemma 4 to obtain

$$\begin{aligned} \frac{n}{2} \cdot \pi &= \int_{x_{n/2}}^{x_n} \theta'(t) dt = \int_0^{x_n} f(t) dt + \sum_{i=n/2}^{n-1} \int_{x_i}^{x_{i+1}} v(t) \cdot \sin(2\theta(t)) dt \\ &< \int_0^{x_n} f(t) dt. \end{aligned} \quad (102)$$

If  $n$  is odd, then we combine (75), (78), (79) in Lemma 2 with (93), (94) in Lemma 4 to obtain

$$\begin{aligned} \frac{n}{2} \cdot \pi &= \int_{t_{(n+1)/2}}^{x_n} \theta'(t) dt = \int_0^{x_n} f(t) dt + \\ &\quad \int_{t_{(n+1)/2}}^{x_{(n+1)/2}} v(t) \cdot \sin(2\theta(t)) dt + \sum_{i=(n+1)/2}^{n-1} \int_{x_i}^{x_{i+1}} v(t) \cdot \sin(2\theta(t)) dt \\ &< \int_0^{x_n} f(t) dt. \end{aligned} \quad (103)$$

We combine (102) and (103) with (61) in Lemma 1 to conclude both (99) and (100).  $\blacksquare$

To prove Theorem 9, we need to develop a number of technical tools. In the following two lemmas, we describe several properties of the equation  $f(t) = v(t)$  in the unknown  $t$ , where  $f, v$  are defined, respectively, via (43), (44) in Section 2.4.

**Lemma 5.** *Suppose that  $n \geq 0$  is a non-negative integer. Suppose also that the functions  $f, v$  are defined, respectively, via (43), (44) in Section 2.4. Suppose furthermore that the real number  $x_n$  is that of Definition 1 in Section 4.1.1. Then, there exists a unique point  $\hat{t}$  in the interval  $(0, x_n)$  such that*

$$f(\hat{t}) = v(\hat{t}). \quad (104)$$

*Proof.* We observe that, due to (43), (44) in Section 2.4,

$$\frac{v(t)}{f(t)} > 0 \quad (105)$$

for all  $0 < t < x_n$ . Moreover,

$$\frac{v(0)}{f(0)} = 0, \quad \lim_{t \rightarrow x_n, t < x_n} \frac{v(t)}{f(t)} = \infty. \quad (106)$$

We combine (82) in the proof of Lemma 3 with (105) and (106) to conclude both existence and uniqueness of the solution to the equation  $f(t) = v(t)$  in the unknown  $0 < t < x_n$ . ■

**Lemma 6.** *Suppose that  $n \geq 2$  is a positive integer. Suppose also that the real number  $x_n$  and the function  $\theta : [-x_n, x_n] \rightarrow \mathbb{R}$  are those of Lemma 2 above. Suppose furthermore that the point  $0 < \hat{t} < x_n$  is that of Lemma 5 above. Then,*

$$\left(n - \frac{1}{4}\right) \cdot \pi < \theta(\hat{t}) < n \cdot \pi. \quad (107)$$

*Proof.* Suppose that the point  $0 < x < x_n$  is defined via the formula

$$x = \theta^{-1} \left( \left(n - \frac{1}{4}\right) \cdot \pi \right), \quad (108)$$

where  $\theta^{-1}$  denotes the inverse of  $\theta$ . By contradiction, suppose that (107) does not hold. In other words,

$$0 < \hat{t} < x. \quad (109)$$

It follows from the combination of Lemma 5, (82) in the proof of Lemma 3, and (109), that  $f(x) < v(x)$ . On the other hand, due to (75) in Lemma 2 and (108),

$$\begin{aligned} \theta'(x) &= f(x) + v(x) \cdot \sin(2\theta(x)) \\ &= f(x) + v(x) \cdot \sin\left(2n\pi - \frac{\pi}{2}\right) = f(x) - v(x) < 0, \end{aligned} \quad (110)$$

in contradiction to (81) in Lemma 3. ■

In the following three lemmas, we study some of the properties of the ratio  $f/v$ , where  $f, v$  are defined, respectively, via (43), (44) in Section 2.4.

**Lemma 7.** *Suppose that  $n \geq 0$  is a non-negative integer, and that the functions  $f, v$  are defined, respectively, via (43), (44) in Section 2.4. Then, for all real  $0 < t < 1$ ,*

$$-\frac{d}{dt} \left( \frac{f}{v} \right) (t) = h_t(a) \cdot f(t), \quad (111)$$

where the real number  $a > 0$  is defined via the formula

$$a = \frac{\chi_n}{c^2}, \quad (112)$$

and, for all  $0 < t < 1$ , the function  $h_t : (0, \infty) \rightarrow \mathbb{R}$  is defined via the formula

$$h_t(a) = \frac{4t^6 + (2a - 6) \cdot t^4 + (4 - 8a) \cdot t^2 + 2a \cdot (a + 1)}{t^2 \cdot (1 + a - 2t^2)^2}. \quad (113)$$

Moreover, for all real  $0 < t < \min \{\sqrt{a}, 1\}$ ,

$$-\frac{d}{dt} \left( \frac{f}{v} \right) (t) \geq \frac{3}{2} \cdot f(t). \quad (114)$$

*Proof.* The identity (111) is obtained from (43), (44) via straightforward algebraic manipulations. To establish (114), it suffices to show that, for a fixed  $0 < t < 1$ ,

$$\inf_a \{ h_t(a) : t^2 < a < \infty \} \geq \frac{3}{2}. \quad (115)$$

We start with observing that, for all  $0 < t < 1$ ,

$$\lim_{a \rightarrow t^2, a > t^2} h_t(a) = 6, \quad \lim_{a \rightarrow \infty} h_t(a) = \frac{2}{t^2}. \quad (116)$$

Then, we differentiate  $h_t(a)$ , given via (113), with respect to  $a$  to obtain

$$\frac{dh_t}{da}(a) = \frac{2 \cdot (1 - t^2)}{t^2 \cdot (1 + a - 2t^2)^3} \cdot (6t^4 + (a - 9) \cdot t^2 + a + 1). \quad (117)$$

It follows from (116), (117), that if  $t^2 < \hat{a}_t < \infty$  is a local extremum of  $h_t(a)$ , then

$$\hat{a}_t = \frac{-6t^4 + 9t^2 - 1}{t^2 + 1} > t^2, \quad (118)$$

which is possible if and only if  $1 > t^2 > 1/7$ . Then we substitute  $\hat{a}_t$ , given via (118), into (113) to obtain

$$h(t, \hat{a}_t) = \frac{-t^4 + 14t^2 - 1}{8t^4}. \quad (119)$$

It is trivial to verify that

$$\inf_t \left\{ h(t, \hat{a}_t) : \frac{1}{7} < t < 1 \right\} = \lim_{t \rightarrow 1, t > 1} h(t, \hat{a}_t) = \frac{3}{2}. \quad (120)$$

Now (115) follows from the combination of (116), (118), (119) and (120). ■

**Lemma 8.** Suppose that  $n \geq 2$  is a positive integer, and that  $t_n$  is the maximal zero of  $\psi_n$  in the interval  $(-1, 1)$ . Suppose also that the real number  $Z_0$  is defined via the formula

$$Z_0 = \frac{1}{1 + \frac{3\pi}{8}} \approx 0.4591. \quad (121)$$

Then, for all  $0 < t \leq t_n$ ,

$$v(t) < f(t) \cdot Z_0, \quad (122)$$

where the functions  $f, v$  are defined, respectively, via (43), (44) in Section 2.4.

*Proof.* Due to (82) in the proof of Lemma 3, the function  $f/v$  decreases monotonically in the interval  $(0, t_n)$ , and therefore, to prove (122), it suffices to show that

$$\frac{f(t_n)}{v(t_n)} > \frac{1}{Z_0} = 1 + \frac{3\pi}{8}. \quad (123)$$

Suppose that the point  $\hat{t}$  is that of Lemma 5. Suppose also that the real number  $x_n$  and the function  $\theta : [-x_n, x_n] \rightarrow \mathbb{R}$  are those of Lemma 2. Suppose furthermore that the function  $s : [0, n \cdot \pi] \rightarrow [-x_n, x_n]$  is the inverse of  $\theta$ . In other words, for all  $0 \leq \eta \leq n \cdot \pi$ ,

$$\theta(s(\eta)) = \eta. \quad (124)$$

We combine Lemma 2, Lemma 3, Lemma 5, Lemma 6 and Lemma 7 to obtain

$$\begin{aligned} \frac{f(t_n)}{v(t_n)} - 1 &= \left(-\frac{f}{v}\right)(\hat{t}) - \left(-\frac{f}{v}\right)(t_n) = \\ &= \int_{t_n}^{\hat{t}} \frac{d}{dt} \left(-\frac{f}{v}\right)(t) dt = \int_{\theta(t_n)}^{\theta(\hat{t})} \frac{\frac{d}{ds} \left(-\frac{f}{v}\right)(s(\eta)) d\eta}{f(s(\eta)) + v(s(\eta)) \cdot \sin(2\eta)} > \\ &= \int_{(n-1/2)\pi}^{(n-1/4)\pi} \frac{\frac{d}{ds} \left(-\frac{f}{v}\right)(s(\eta)) d\eta}{f(s(\eta)) + v(s(\eta)) \cdot \sin(2\eta)} > \\ &= \int_{(n-1/2)\pi}^{(n-1/4)\pi} \frac{\frac{d}{ds} \left(-\frac{f}{v}\right)(s(\eta)) d\eta}{f(s(\eta))} > \frac{\pi}{4} \cdot \frac{3}{2} = \frac{3\pi}{8}, \end{aligned} \quad (125)$$

which implies (123). ■

**Lemma 9.** Suppose that  $n \geq 2$  and  $(n+1)/2 \leq i \leq n-1$  are positive integers. Suppose also that the real number  $x_n$  and the function  $\theta : [-x_n, x_n] \rightarrow \mathbb{R}$  are those of Lemma 2. Suppose furthermore that  $0 < \delta < \pi/4$  is a real number, and that the real number  $Z_\delta$  is defined via the formula

$$Z_\delta = \left[1 + \frac{3}{2} \cdot \left(\frac{\pi}{4} + \frac{\delta}{1 + Z_0 \cdot \sin(2\delta)}\right)\right]^{-1}, \quad (126)$$

where  $Z_0$  is defined via (121) in Lemma 8 above. Then,

$$v(t) < f(t) \cdot Z_\delta, \quad (127)$$

for all  $0 < t \leq s((i+1/2) \cdot \pi - \delta)$ , where the functions  $f, v$  are defined, respectively, via (43), (44) in Section 2.4, and the function  $s : [0, n \cdot \pi] \rightarrow [-x_n, x_n]$  is the inverse of  $\theta$ .

*Proof.* Suppose that the point  $t_\delta$  is defined via the formula

$$t_\delta = s((i + 1/2) \cdot \pi - \delta). \quad (128)$$

Due to (82) in the proof of Lemma 3, the function  $f/v$  decreases monotonically in the interval  $(0, t_\delta)$ , and therefore to prove (127) it suffices to show that

$$\frac{f(t_\delta)}{v(t_\delta)} > \frac{1}{Z_\delta} = 1 + \frac{3}{2} \cdot \left( \frac{\pi}{4} + \frac{\delta}{1 + Z_0 \cdot \sin(2\delta)} \right). \quad (129)$$

We observe that, due to Lemma 3,

$$0 \leq \sin(2\theta(t)) \leq \sin(2\delta), \quad (130)$$

for all  $t_\delta \leq t \leq s((i + 1/2)\pi)$ . We combine (128), (130) with Lemma 2, Lemma 3, Lemma 6, Lemma 7 and Lemma 8 to obtain

$$\begin{aligned} \frac{f(t_\delta)}{v(t_\delta)} - \frac{f(s((i + 1/2)\pi))}{v(s((i + 1/2)\pi))} &= \\ \int_{t_\delta}^{s((i+1/2)\pi)} \frac{d}{dt} \left( -\frac{f}{v} \right) (t) dt &= \int_{(i-1/2)\pi-\delta}^{(i-1/2)\pi} \frac{\frac{d}{dt} \left( -\frac{f}{v} \right) (s(\eta)) d\eta}{f(s(\eta)) + v(s(\eta)) \cdot \sin(2\eta)} > \\ \int_{(i-1/2)\pi-\delta}^{(i-1/2)\pi} \frac{\frac{d}{dt} \left( -\frac{f}{v} \right) (s(\eta))}{f(s(\eta))} \cdot \frac{d\eta}{1 + (v/f)(s(\eta)) \cdot \sin(2\delta)} &> \\ \frac{3}{2} \cdot \delta \cdot \frac{1}{1 + Z_0 \cdot \sin(2\delta)}. \end{aligned} \quad (131)$$

We combine (131) with (122) in Lemma 8 to obtain (129), which, in turn, implies (127).  $\blacksquare$

In the following two lemmas, we estimate the rate of decay of the ratio  $f/v$  and its relationship with  $\theta$  of the ODE (75) in Lemma 2.

**Lemma 10.** *Suppose that  $n \geq 2$  and  $(n + 1)/2 \leq i \leq n - 1$  are positive integers. Suppose also that the real number  $x_n$  and the function  $\theta : [-x_n, x_n] \rightarrow \mathbb{R}$  are those of Lemma 2. Suppose furthermore that  $0 < \delta < \pi/4$  is a real number. Then,*

$$\left( \frac{f}{v} \right) (s(i\pi - \delta)) - \left( \frac{f}{v} \right) (s(i\pi - \delta + \pi/2)) > 2 \cdot \sin(2\delta), \quad (132)$$

where the functions  $f, v$  are defined, respectively, via (43), (44) in Section 2.4, and the function  $s : [0, n \cdot \pi] \rightarrow [-x_n, x_n]$  is the inverse of  $\theta$ .

*Proof.* We observe that, due to Lemma 2 and Lemma 3,

$$\sin(2\theta(t)) > 0, \quad (133)$$

for all  $s(i\pi) < t < s(i\pi - \delta + \pi/2)$ . We combine (133) with Lemma 2, Lemma 3, Lemma 6, Lemma 7 and Lemma 9 to obtain

$$\begin{aligned} \frac{f(s(i\pi))}{v(s(i\pi))} - \frac{f(s(i\pi - \delta + \pi/2))}{v(s(i\pi - \delta + \pi/2))} &= \\ \int_{s(i\pi)}^{s(i\pi - \delta + \pi/2)} \frac{d}{dt} \left( -\frac{f}{v} \right) (t) dt &= \int_{i\pi}^{i\pi - \delta + \pi/2} \frac{\frac{d}{dt} \left( -\frac{f}{v} \right) (s(\eta)) d\eta}{f(s(\eta)) + v(s(\eta)) \cdot \sin(2\eta)} > \\ \int_{i\pi}^{i\pi - \delta + \pi/2} \frac{\frac{d}{dt} \left( -\frac{f}{v} \right) (s(\eta))}{f(s(\eta))} \cdot \frac{d\eta}{1 + (v/f)(s(\eta))} &> \frac{3}{2} \cdot \left( \frac{\pi}{2} - \delta \right) \cdot \frac{1}{1 + Z_\delta}, \end{aligned} \quad (134)$$

where  $Z_\delta$  is defined via (126) in Lemma 9. We also observe that, due to Lemma 2 and Lemma 3,

$$\sin(2\theta(t)) < 0, \quad (135)$$

for all  $s(i\pi - \delta) < t < s(i\pi)$ . We combine (135) with Lemma 2, Lemma 3, Lemma 6 and Lemma 7 to obtain

$$\begin{aligned} \frac{f(s(i\pi - \delta))}{v(s(i\pi - \delta))} - \frac{f(s(i\pi))}{v(s(i\pi))} &= \\ \int_{s(i\pi - \delta)}^{s(i\pi)} \frac{d}{dt} \left( -\frac{f}{v} \right) (t) dt &= \int_{i\pi - \delta}^{i\pi} \frac{\frac{d}{dt} \left( -\frac{f}{v} \right) (s(\eta)) d\eta}{f(s(\eta)) + v(s(\eta)) \cdot \sin(2\eta)} > \\ \int_{i\pi - \delta}^{i\pi} \frac{\frac{d}{dt} \left( -\frac{f}{v} \right) (s(\eta))}{f(s(\eta))} d\eta &> \frac{3}{2} \cdot \delta. \end{aligned} \quad (136)$$

Next, suppose that the function  $h : [0, \pi/4] \rightarrow \mathbb{R}$  is defined via the formula

$$h(\delta) = \frac{3}{2} \cdot \left( \frac{\pi}{2} - \delta \right) \cdot \frac{1}{1 + Z_\delta} + \frac{3}{2} \cdot \delta - 2 \cdot \sin(2\delta), \quad (137)$$

where  $Z_\delta$  is defined via (126) in Lemma 9. One can easily verify that

$$\min_{\delta} \{h(\delta) : 0 \leq \delta \leq \pi/4\} > \frac{1}{25}, \quad (138)$$

and, in particular, that  $h(\delta) > 0$  for all  $0 \leq \delta \leq \pi/4$ . We combine (134), (136), (137) and (138) to obtain (132).  $\blacksquare$

**Lemma 11.** *Suppose that  $n \geq 2$  and  $(n+1)/2 \leq i \leq n-1$  are positive integers. Suppose also that the real number  $x_n$  and the function  $\theta : [-x_n, x_n] \rightarrow \mathbb{R}$  are those of Lemma 2. Suppose furthermore that  $0 < \delta < \pi/4$  is a real number. Then,*

$$\left( \frac{f}{v} \right) (s(i\pi + \delta - \pi/2)) - \left( \frac{f}{v} \right) (s(i\pi + \delta)) > 2 \cdot \sin(2\delta), \quad (139)$$

where the functions  $f, v$  are defined, respectively, via (43), (44) in Section 2.4, and the function  $s : [0, n \cdot \pi] \rightarrow [-x_n, x_n]$  is the inverse of  $\theta$ .



*Proof.* We observe that, due to Lemma 2 and Lemma 3,

$$\sin(2\theta(t)) > 0, \quad (140)$$

for all  $s(i\pi) < t < s(i\pi + \delta)$ . We combine (140) with Lemma 2, Lemma 3, Lemma 6, Lemma 7 and Lemma 9 to obtain

$$\begin{aligned} \frac{f(s(i\pi))}{v(s(i\pi))} - \frac{f(s(i\pi + \delta))}{v(s(i\pi + \delta))} &= \\ \int_{s(i\pi)}^{s(i\pi + \delta)} \frac{d}{dt} \left( -\frac{f}{v} \right) (t) dt &= \int_{i\pi}^{i\pi + \delta} \frac{\frac{d}{dt} \left( -\frac{f}{v} \right) (s(\eta)) d\eta}{f(s(\eta)) + v(s(\eta)) \cdot \sin(2\eta)} > \\ \int_{i\pi}^{i\pi + \delta} \frac{\frac{d}{dt} \left( -\frac{f}{v} \right) (s(\eta))}{f(s(\eta))} \cdot \frac{d\eta}{1 + (v/f)(s(\eta))} &> \frac{3}{2} \cdot \frac{\delta}{1 + Z_\delta}, \end{aligned} \quad (141)$$

where  $Z_\delta$  is defined via (126) in Lemma 9. We also observe that, due to Lemma 2 and Lemma 3,

$$\sin(2\theta(t)) < 0, \quad (142)$$

for all  $s(i\pi + \delta - \pi/2) < t < s(i\pi)$ . We combine (142) with Lemma 2, Lemma 3, Lemma 6 and Lemma 7 to obtain

$$\begin{aligned} \frac{f(s(i\pi + \delta - \pi/2))}{v(s(i\pi + \delta - \pi/2))} - \frac{f(s(i\pi))}{v(s(i\pi))} &= \\ \int_{s(i\pi + \delta - \pi/2)}^{s(i\pi)} \frac{d}{dt} \left( -\frac{f}{v} \right) (t) dt &= \int_{i\pi + \delta - \pi/2}^{i\pi} \frac{\frac{d}{dt} \left( -\frac{f}{v} \right) (s(\eta)) d\eta}{f(s(\eta)) + v(s(\eta)) \cdot \sin(2\eta)} > \\ \int_{i\pi + \delta - \pi/2}^{i\pi} \frac{\frac{d}{dt} \left( -\frac{f}{v} \right) (s(\eta)) d\eta}{f(s(\eta))} &> \frac{3}{2} \cdot \left( \frac{\pi}{2} - \delta \right). \end{aligned} \quad (143)$$

Obviously, for all  $0 < \delta < \pi/4$ ,

$$\frac{3}{2} \cdot \frac{\delta}{1 + Z_\delta} + \frac{3}{2} \cdot \left( \frac{\pi}{2} - \delta \right) > \frac{3}{2} \cdot \left( \frac{\pi}{2} - \delta \right) \cdot \frac{1}{1 + Z_\delta} + \frac{3}{2} \cdot \delta. \quad (144)$$

We combine (141), (143), (144) with (137), (138) in the proof of Lemma 10 to obtain (139). ■

In the following lemma, we analyze the integral of the oscillatory part of the right-hand side of the ODE (75) between consecutive roots of  $\psi_n$ . This lemma can be viewed as an extension of Lemma 4, and is used in the proof of Theorem 9 below.

**Lemma 12.** *Suppose that  $n \geq 2$  is an integer,  $-1 < t_1 < t_2 < \dots < t_n < 1$  are the roots of  $\psi_n$  in the interval  $(-1, 1)$ , and  $x_1 < \dots < x_{n-1}$  are the roots of  $\psi'_n$  in the interval  $(t_1, t_n)$ . Suppose also, that the real number  $x_n$  and the function  $\theta : [-x_n, x_n] \rightarrow \mathbb{R}$  are those of*

Lemma 2 above. Suppose furthermore that the function  $v$  is defined via (44) in Section 2.4. Then,

$$\int_{t_i}^{t_{i+1}} v(t) \cdot \sin(2\theta(t)) dt > 0, \quad (145)$$

for all integer  $(n+1)/2 \leq i \leq n-1$ , i.e. for all integer  $i$  such that  $0 \leq t_i < t_n$ .

*Proof.* Suppose that  $i$  is a positive integer such that  $(n-1)/2 \leq i \leq n-1$ . Suppose also that the function  $s : [0, n \cdot \pi] \rightarrow [-x_n, x_n]$  is the inverse of  $\theta$ . In other words, for all  $0 \leq \eta \leq n \cdot \pi$ ,

$$\theta(s(\eta)) = \eta. \quad (146)$$

Due to (97) in the proof of Lemma 4 above,

$$\begin{aligned} & \int_{t_i}^{x_i} v(t) \cdot \sin(2\theta(t)) dt = \\ & - \int_0^{\pi/2} \frac{v(s(i\pi + \eta - \pi/2)) \cdot \sin(2\eta) d\eta}{f(s(i\pi + \eta - \pi/2)) - v(s(i\pi + \eta - \pi/2)) \cdot \sin(2\eta)}. \end{aligned} \quad (147)$$

We proceed to compare the integrand in (147) to the integrand in (96) in the proof of Lemma 4. First, for all  $0 < \eta < \pi/4$ ,

$$\frac{1}{(f/v)(s(i\pi + \eta - \pi/2)) - \sin(2\eta)} < \frac{1}{(f/v)(s(i\pi + \eta)) + \sin(2\eta)}, \quad (148)$$

due to (139) in Lemma 11. Moreover, for all  $\pi/4 < \eta < \pi/2$ , we substitute  $\delta = \pi/2 - \eta$  to obtain

$$\begin{aligned} \frac{1}{(f/v)(s(i\pi + \eta - \pi/2)) - \sin(2\eta)} &= \frac{1}{(f/v)(s(i\pi - \delta)) - \sin(2\delta)} < \\ \frac{1}{(f/v)(s(i\pi - \delta + \pi/2)) + \sin(2\delta)} &= \frac{1}{(f/v)(s(i\pi + \eta)) + \sin(2\eta)}, \end{aligned} \quad (149)$$

due to (132) in Lemma 10. We combine (96) in the proof of Lemma 4 with (147), (148), (149) to obtain (145).  $\blacksquare$

The following theorem is a counterpart of Theorem 8 above. It is illustrated in Tables 1, 2, 3, 4.

**Theorem 9.** Suppose that  $n \geq 2$  is a positive integer. Suppose also that  $t_n$  is the maximal root of  $\psi_n$  in  $(-1, 1)$ . Then,

$$1 + \frac{2}{\pi} \int_0^{t_n} \sqrt{\frac{\chi_n - c^2 t^2}{1 - t^2}} dt < n. \quad (150)$$

*Proof.* Suppose that the real numbers

$$-1 \leq -x_n < t_1 < x_1 < t_2 < \cdots < t_{n-1} < x_{n-1} < t_n < x_n \leq 1 \quad (151)$$

and the function  $\theta : [-x_n, x_n] \rightarrow \mathbb{R}$  are those of Lemma 2 above. Suppose also that the functions  $f, v$  are defined, respectively, via (43),(44) in Section 2.4. If  $n$  is odd, then we combine (75), (78), in Lemma 2 with (145) in Lemma 12 to obtain

$$\begin{aligned} \frac{n-1}{2} \cdot \pi &= \int_{t_{(n+1)/2}}^{t_n} \theta'(t) dt = \int_0^{t_n} f(t) dt + \sum_{i=(n+1)/2}^{n-1} \int_{t_i}^{t_{i+1}} v(t) \cdot \sin(2\theta(t)) dt \\ &> \int_0^{t_n} f(t) dt. \end{aligned} \quad (152)$$

If  $n$  is even, then we combine (75), (78), (79) in Lemma 2 with (92) in Lemma 4 and (145) in Lemma 12 to obtain

$$\begin{aligned} \frac{n-1}{2} \cdot \pi &= \int_{x_{n/2}}^{t_n} \theta'(t) dt = \int_0^{t_n} f(t) dt + \\ &\quad \int_{x_{n/2}}^{t_{(n/2)+1}} v(t) \cdot \sin(2\theta(t)) dt + \sum_{i=(n/2)+1}^{n-1} \int_{t_i}^{t_{i+1}} v(t) \cdot \sin(2\theta(t)) dt \\ &> \int_0^{t_n} f(t) dt. \end{aligned} \quad (153)$$

We combine (152) and (153) to conclude (150).  $\blacksquare$

**Corollary 2.** Suppose that  $n \geq 2$  is a positive integer, and that  $\chi_n > c^2$ . Suppose also that  $t_n$  is the maximal root of  $\psi_n$  in the interval  $(-1, 1)$ . Then,

$$1 + \frac{2}{\pi} \sqrt{\chi_n} \cdot E\left(\operatorname{asin}(t_n), \frac{c}{\sqrt{\chi_n}}\right) < n < \frac{2}{\pi} \sqrt{\chi_n} \cdot E\left(\frac{c}{\sqrt{\chi_n}}\right), \quad (154)$$

where  $E(y, k)$  and  $E(k)$  are defined, respectively, via (18) and (22) in Section 2.2.

*Proof.* It follows immediately from (20), (99) in Theorem 8 and (150) in Theorem 9.  $\blacksquare$

The following theorem, illustrated in Tables 7, 8, provides upper and lower bounds on the distance between consecutive roots of  $\psi_n$  inside  $(-1, 1)$ .

**Theorem 10.** Suppose that  $n \geq 2$  is a positive integer, and that  $\chi_n > c^2$ . Suppose also that  $-1 < t_1 < t_2 < \cdots < t_n < 1$  are the roots of  $\psi_n$  in the interval  $(-1, 1)$ . Suppose furthermore that the functions  $f, v$  are defined, respectively, via (43),(44) in Section 2.4. Then,

$$\frac{\pi}{f(t_{i+1}) + v(t_{i+1})/2} < t_{i+1} - t_i < \frac{\pi}{f(t_i)}, \quad (155)$$

for all integer  $(n+1)/2 \leq i \leq n-1$ , i.e. for all integer  $i$  such that  $0 \leq t_i < t_n$ .

*Proof.* Suppose that the function  $\theta : [-1, 1] \rightarrow \mathbb{R}$  is that of Lemma 2. We observe that  $f$  is increasing in  $(0, 1)$  due to (43) in Section 2.4, and combine this observation with (75), (78), in Lemma 2 and (145) in Lemma 12 to obtain

$$\begin{aligned} \pi &= \int_{t_i}^{t_{i+1}} \theta'(t) dt = \int_{t_i}^{t_{i+1}} f(t) dt + \int_{t_i}^{t_{i+1}} v(t) \cdot \sin(2\theta(t)) dt \\ &> \int_{t_i}^{t_{i+1}} f(t) dt > (t_{i+1} - t_i) \cdot f(t_i), \end{aligned} \quad (156)$$

which implies the right-hand side of (155). As in Lemma 2, suppose that  $x_i$  is the zero of  $\psi'_n$  in the interval  $(t_i, t_{i+1})$ . We combine (75), (78), (78) in Lemma 2 and (92), (93) in Lemma 4 to obtain

$$\int_{t_i}^{x_i} f(t) dt > \int_{t_i}^{x_i} \theta'(t) dt = \pi = \int_{x_i}^{t_{i+1}} \theta'(t) dt > \int_{x_i}^{t_{i+1}} f(t) dt. \quad (157)$$

Since  $f$  is increasing in  $(t_i, t_{i+1})$  due to (43) in Section 2.4, the inequality (157) implies that

$$x_i - t_i > t_{i+1} - x_i. \quad (158)$$

Moreover, we observe that  $v$  is also increasing in  $(0, 1)$ . We combine this observation with (158), (75), (78), in Lemma 2 and (92), (93) in Lemma 4 to obtain

$$\begin{aligned} \pi &= \int_{t_i}^{t_{i+1}} \theta'(t) dt < \int_{t_i}^{t_{i+1}} f(t) dt + \int_{x_i}^{t_{i+1}} v(t) \cdot \sin(2\theta(t)) dt \\ &< (t_{i+1} - t_i) \cdot f(t_{i+1}) + (t_{i+1} - x_i) \cdot v(t_{i+1}) \\ &< (t_{i+1} - t_i) \cdot f(t_{i+1}) + \frac{t_{i+1} - t_i}{2} \cdot v(t_{i+1}), \end{aligned} \quad (159)$$

which implies the left-hand side of (155). ■

The following theorem is a direct consequence of Theorem 8 above.

**Theorem 11.** *Suppose that  $n \geq 2$  is a positive integer. If  $n \geq 2c/\pi$ , then  $\chi_n > c^2$ .*

*Proof.* Suppose that  $\chi_n < c^2$ , and  $T$  is the maximal root of  $\psi'_n$  in  $(0, 1)$ , as in Theorem 8 above. Then,

$$n < \frac{2}{\pi} \int_0^T \sqrt{\frac{\chi_n - c^2 t^2}{1 - t^2}} dt = \frac{2c}{\pi} \int_0^T \sqrt{\frac{\chi_n/c^2 - t^2}{1 - t^2}} dt < \frac{2c}{\pi} \cdot T < \frac{2c}{\pi}, \quad (160)$$

due to (100) in Theorem 8. ■

#### 4.1.3 Elimination of the First-Order Term of the Prolate ODE

In this subsection, we analyze the oscillation properties of  $\psi_n$  via transforming the ODE (14) into a second-order linear ODE without the first-order term. The following lemma is the principal technical tool of this subsection.

**Lemma 13.** Suppose that  $n \geq 0$  is a non-negative integer. Suppose also that the functions  $\Psi_n, Q_n : (-1, 1) \rightarrow \mathbb{R}$  are defined, respectively, via the formulae

$$\Psi_n(t) = \psi_n(t) \cdot \sqrt{1 - t^2} \quad (161)$$

and

$$Q_n(t) = \frac{\chi_n - c^2 \cdot t^2}{1 - t^2} + \frac{1}{(1 - t^2)^2}, \quad (162)$$

for  $-1 < t < 1$ . Then,

$$\Psi_n''(t) + Q_n(t) \cdot \Psi_n(t) = 0, \quad (163)$$

for all  $-1 < t < 1$ .

*Proof.* We differentiate  $\Psi_n$  with respect to  $t$  to obtain

$$\Psi_n'(t) = \psi_n'(t) \sqrt{1 - t^2} - \psi_n(t) \cdot \frac{t}{\sqrt{1 - t^2}}. \quad (164)$$

Then, using (164), we differentiate  $\Psi_n'$  with respect to  $t$  to obtain

$$\begin{aligned} \Psi_n''(t) &= \psi_n''(t) \sqrt{1 - t^2} - \psi_n'(t) \cdot \frac{2t}{\sqrt{1 - t^2}} - \psi_n(t) \cdot \frac{\sqrt{1 - t^2} + t^2/\sqrt{1 - t^2}}{1 - t^2} \\ &= \psi_n''(t) \sqrt{1 - t^2} - \psi_n'(t) \cdot \frac{2t}{\sqrt{1 - t^2}} - \psi_n(t) (1 - t^2)^{-\frac{3}{2}} \\ &= \frac{1}{\sqrt{1 - t^2}} \left[ (1 - t^2) \cdot \psi_n''(t) - 2t \cdot \psi_n'(t) - \frac{\psi_n(t)}{1 - t^2} \right] \\ &= \frac{1}{\sqrt{1 - t^2}} \left[ -\psi_n(t) \cdot (\chi_n - c^2 \cdot t^2) - \frac{\psi_n(t)}{1 - t^2} \right] \\ &= -\Psi_n(t) \cdot \left( \frac{\chi_n - c^2 \cdot t^2}{1 - t^2} + \frac{1}{(1 - t^2)^2} \right). \end{aligned} \quad (165)$$

We observe that (163) follows from (165). ■

In the next theorem, we provide an upper bound on  $\chi_n$  in terms of  $n$ . The results of the corresponding numerical experiments are reported in Tables 5, 6.

**Theorem 12.** Suppose that  $n \geq 2$  is a positive integer, and that  $\chi_n > c^2$ . Then,

$$\chi_n < \left( \frac{\pi}{2} (n + 1) \right)^2. \quad (166)$$

*Proof.* Suppose that the functions  $\Psi_n, Q_n : (-1, 1) \rightarrow \mathbb{R}$  are those of Lemma 13 above. We observe that, since  $\chi_n > c^2$ ,

$$Q_n(t) > \chi_n + 1, \quad (167)$$

for  $-1 < t < 1$ . Suppose now that  $t_n$  is the maximal root of  $\psi_n$  in  $(-1, 1)$ . We combine (167) with (163) in Lemma 13 above and Theorem 7, Corollary 1 in Section 2.3 to obtain the inequality

$$t_n \geq 1 - \frac{\pi}{\sqrt{\chi_n + 1}}. \quad (168)$$

Then, we combine (168) with Theorem 9 above to obtain

$$\begin{aligned} n &> 1 + \frac{2}{\pi} \int_0^{t_n} \sqrt{\frac{\chi_n - c^2 t^2}{1 - t^2}} dt \\ &> 1 + \frac{2 \cdot t_n}{\pi} \sqrt{\chi_n} \geq 1 + \frac{2}{\pi} \sqrt{\chi_n} \left(1 - \frac{\pi}{\sqrt{\chi_n + 1}}\right) > \frac{2}{\pi} \sqrt{\chi_n} - 1, \end{aligned} \quad (169)$$

which implies (166). ■

The following theorem is a consequence of the proof of Theorem 12.

**Theorem 13.** *Suppose that  $n \geq 2$  is a positive integer, and that  $\chi_n > c^2$ . Suppose also that  $t_1 < \dots < t_n$  are the roots of  $\psi_n$  in  $(-1, 1)$ . Then,*

$$t_{j+1} - t_j < \frac{\pi}{\sqrt{\chi_n + 1}}, \quad (170)$$

for all  $j = 1, 2, \dots, n-1$ .

*Proof.* The inequality (170) follows from the combination of (167) in the proof of Theorem 12, (163) in Lemma 13 and Theorem 7, Corollary 1 in Section 2.3. ■

The following theorem extends Theorem 11 in Section 4.1.2.

**Theorem 14.** *Suppose that  $n \geq 2$  is a positive integer.*

- *If  $n \leq (2c/\pi) - 1$ , then  $\chi_n < c^2$ .*
- *If  $n \geq (2c/\pi)$ , then  $\chi_n > c^2$ .*
- *If  $(2c/\pi) - 1 < n < (2c/\pi)$ , then either inequality is possible.*

*Proof.* Suppose that  $\chi_n > c^2$ , and that the functions  $\Psi_n, Q_n : (-1, 1) \rightarrow \mathbb{R}$  are those of Lemma 13 above. Suppose also that  $t_1 < \dots < t_n$  are the roots of  $\psi_n$  in  $(-1, 1)$ . We observe that, due to (162) in Lemma 13,

$$Q_n(t) = c^2 + \frac{\chi_n - c^2}{1 - t^2} + \frac{1}{(1 - t^2)^2} > c^2. \quad (171)$$

We combine (171) with (163) in Lemma 13 above and Theorem 5 in Section 2.3 to conclude that

$$t_{j+1} - t_j < \frac{\pi}{c}, \quad (172)$$

for all  $j = 1, \dots, n-1$ , and, moreover,

$$1 - t_n < \frac{\pi}{c}. \quad (173)$$

We combine (172) with (173) to obtain the inequality

$$2 \left(1 - \frac{\pi}{c}\right) < 2t_n = t_n - t_1 < (n-1) \frac{\pi}{c}, \quad (174)$$

which implies that

$$n > \frac{2}{\pi}c - 1. \quad (175)$$

We conclude the proof by combining Theorem 11 in Section 4.1.2 with (175).  $\blacksquare$

The following theorem is yet another application of Lemma 13 above.

**Theorem 15.** *Suppose that  $n \geq 2$  is a positive integer. Suppose also that  $-1 < t_1 < t_2 < \dots < t_n < 1$  are the roots of  $\psi_n$  in the interval  $(-1, 1)$ . Suppose furthermore that  $i$  is an integer such that  $0 \leq t_i < t_n$ , i.e.  $(n+1)/2 \leq i \leq n-1$ . If  $\chi_n > c^2$ , then*

$$t_{i+1} - t_i > t_{i+2} - t_{i+1} > \dots > t_n - t_{n-1}. \quad (176)$$

*If  $\chi_n < c^2 - c\sqrt{2}$ , then*

$$t_{i+1} - t_i < t_{i+2} - t_{i+1} < \dots < t_n - t_{n-1}. \quad (177)$$

*Proof.* Suppose that the functions  $\Psi_n, Q_n : (-1, 1) \rightarrow \mathbb{R}$  are those of Lemma 13 above. If  $\chi_n > c^2$ , then, due to (162) in Lemma 13,

$$Q_n(t) = c^2 + \frac{\chi_n - c^2}{1 - t^2} + \frac{1}{(1 - t^2)^2} \quad (178)$$

is obviously a monotonically increasing function. We combine this observation with (163) of Lemma 13 and (28) of Theorem 6 in Section 2.3 to conclude (176).

Suppose now that

$$\chi_n < c^2 - c\sqrt{2}. \quad (179)$$

Suppose also that the function  $P_n : (1, \infty) \rightarrow \mathbb{R}$  is defined via the formula

$$P_n(y) = Q_n\left(\sqrt{1 - \frac{1}{\sqrt{y}}}\right) = y^2 + (\chi_n - c^2) \cdot y + c^2, \quad (180)$$

for  $1 < y < \infty$ . Obviously,

$$Q_n(t) = P_n\left(\frac{1}{1 - t^2}\right). \quad (181)$$

Suppose also that  $y_0$  is defined via the formula

$$y_0 = \frac{1}{1 - (\sqrt{\chi_n}/c)^2} = \frac{c^2}{c^2 - \chi_n}. \quad (182)$$

We combine (179), (180) and (182) to conclude that, for  $1 < y < y_0$ ,

$$P'_n(y) = 2y - (c^2 - \chi_n) < 2y_0 - (c^2 - \chi_n) = \frac{2c^2 - (c^2 - \chi_n)^2}{c^2 - \chi_n} < 0. \quad (183)$$

Moreover, due to (180), (182), (183),

$$P_n(y) > P_n(y_0) = \left( \frac{c^2}{\chi_n - c^2} \right)^2 > 0, \quad (184)$$

for all  $1 < y < y_0$ . We combine (180), (181), (182), (183) and (184) to conclude that  $Q_n$  is monotonically decreasing and strictly positive in the interval  $(0, \sqrt{\chi_n}/c)$ . We combine this observation with (29) of Theorem 6 in Section 2.3, (63) of Lemma 1, and (163) of Lemma 13 to conclude (177).  $\blacksquare$

**Remark 2.** *Numerical experiments confirm that there exist real  $c > 0$  and integer  $n > 0$  such that  $c^2 - c\sqrt{2} < \chi_n < c^2$  and neither of (176), (177) is true.*

In the following theorem, we provide an upper bound on  $1 - t_n$ , where  $t_n$  is the maximal root of  $\psi_n$  in the interval  $(-1, 1)$ .

**Theorem 16.** *Suppose that  $n \geq 2$  is a positive integer, and that  $\chi_n > c^2$ . Suppose also that  $t_n$  is the maximal root of  $\psi_n$  in the interval  $(-1, 1)$ . Then,*

$$c^2 \cdot (1 - t_n)^2 + \frac{\chi_n - c^2}{1 + t_n} \cdot (1 - t_n) < \pi^2. \quad (185)$$

Moreover,

$$1 - t_n < \frac{4\pi^2}{\chi_n - c^2 + \sqrt{(\chi_n - c^2)^2 + (4\pi c)^2}}. \quad (186)$$

*Proof.* Suppose that the functions  $\Psi_n, Q_n : (-1, 1) \rightarrow \mathbb{R}$  are those of Lemma 13 above. Since  $\chi_n > c^2$ , the function  $Q_n$  is monotonically increasing, i.e.

$$Q_n(t_n) \leq Q(t), \quad (187)$$

for all  $t_n \leq t < 1$ . We consider the solution  $\varphi_n$  of the ODE

$$\varphi_n''(t) + Q_n(t_n) \cdot \varphi_n(t) = 0, \quad (188)$$

with the initial conditions

$$\varphi(t_n) = \Psi_n(t_n) = 0, \quad \varphi'(t_n) = \Psi'_n(t_n). \quad (189)$$



The function  $\varphi_n$  has a root  $y_n$  given via the formula

$$y_n = t_n + \frac{\pi}{\sqrt{Q_n(t_n)}}. \quad (190)$$

Suppose, by contradiction, that  $y_n \leq 1$ . Then, due to the combination of (163) of Lemma 13, Theorem 7, Corollary 1 in Section 2.3, and (187) above,  $\Psi_n$  has a root in the interval  $(t_n, y_n)$ , in contradiction to (161). Therefore,

$$t_n + \frac{\pi}{\sqrt{Q_n(t_n)}} > 1. \quad (191)$$

We rewrite (191) as

$$(1 - t_n)^2 \cdot Q_n(t_n) < \pi^2, \quad (192)$$

and plug (162) into (192) to obtain the inequality

$$c^2 \cdot (1 - t_n)^2 + \frac{\chi_n - c^2}{1 + t_n} \cdot (1 - t_n) + \frac{1}{(1 + t_n)^2} < \pi^2, \quad (193)$$

which immediately yields (185). Since  $1 - t_n$  is positive, (193) implies that  $1 - t_n$  is bounded from above by the maximal root  $x_{\max}$  of the quadratic equation

$$c^2 \cdot x^2 + \frac{\chi_n - c^2}{2} \cdot x - \pi^2 = 0, \quad (194)$$

given via the formula

$$\begin{aligned} x_{\max} &= \frac{1}{4c^2} \cdot \left( \sqrt{(\chi_n - c^2)^2 + 16\pi^2 c^2} - (\chi_n - c^2) \right) \\ &= \frac{16\pi^2 c^2}{4c^2} \cdot \frac{1}{\chi_n - c^2 + \sqrt{(\chi_n - c^2)^2 + 16\pi^2 c^2}}, \end{aligned} \quad (195)$$

which implies (186). ■

The following theorem uses Theorem 16 to simplify the inequalities (99) in Theorem 8 and (150) in Theorem 9 in Section 4.1.2.

**Theorem 17.** *Suppose that  $n \geq 2$  is a positive integer, and that  $\chi_n > c^2$ . Suppose also that  $t_n$  is the maximal root of  $\psi_n$  in the interval  $(-1, 1)$ . Then,*

$$\frac{2}{\pi} \int_{t_n}^1 \sqrt{\frac{\chi_n - c^2 t^2}{1 - t^2}} dt < 4. \quad (196)$$

Moreover,

$$n < \frac{2}{\pi} \int_0^1 \sqrt{\frac{\chi_n - c^2 t^2}{1 - t^2}} dt < n + 3. \quad (197)$$

*Proof.* We observe that, for  $t_n \leq t < 1$ ,

$$\frac{\chi_n - c^2 t^2}{1+t} < \frac{\chi_n - c^2 t_n^2}{1+t_n} = \frac{\chi_n - c^2}{1+t_n} + \frac{c^2 - c^2 t_n^2}{1+t_n} = c^2(1-t_n) + \frac{\chi_n - c^2}{1+t_n}. \quad (198)$$

We combine (198) with (185) in Theorem 16 to obtain the inequality

$$\frac{\chi_n - c^2 t^2}{1+t} < \frac{\pi^2}{1-t_n}, \quad (199)$$

valid for  $t_n \leq t < 1$ . We conclude from (199) that

$$\int_{t_n}^1 \sqrt{\frac{\chi_n - c^2 t^2}{1-t^2}} dt < \frac{\pi}{\sqrt{1-t_n}} \cdot \int_{t_n}^1 \frac{dt}{\sqrt{1-t}} = \frac{\pi}{\sqrt{1-t_n}} \cdot 2\sqrt{1-t_n} = 2\pi, \quad (200)$$

which implies (196). The inequality (197) follows from the combination of (196), (99) in Theorem 8 and (150) in Theorem 9 in Section 4.1.2.  $\blacksquare$

**Corollary 3.** *Suppose that  $n \geq 2$  is a positive integer, and that  $\chi_n > c^2$ . Then,*

$$n < \frac{2}{\pi} \sqrt{\chi_n} \cdot E\left(\frac{c}{\sqrt{\chi_n}}\right) < n+3, \quad (201)$$

where  $E(k)$  is defined via (22) in Section 2.2.

*Proof.* The inequality (201) follows immediately from the combination of (22) in Section 2.2 and (197) in Theorem 17 above.  $\blacksquare$

The following theorem extends Theorem 16 above by providing a lower bound on  $1-t_n$ , where  $t_n$  is the maximal root of  $\psi_n$  in the interval  $(-1, 1)$ .

**Theorem 18.** *Suppose that  $n \geq 2$  is a positive integer, and that  $\chi_n > c^2$ . Suppose also that  $t_n$  is the maximal root of  $\psi_n$  in the interval  $(-1, 1)$ . Then,*

$$\frac{\pi^2/8}{\chi_n - c^2 + \sqrt{(\chi_n - c^2)^2 + (\pi c/2)^2}} < 1 - t_n. \quad (202)$$

*Proof.* We combine the inequalities (99) in Theorem 8 and (150) in Theorem 9 in Section 4.1.2 to conclude that

$$1 < \frac{2}{\pi} \int_{t_n}^1 \sqrt{\frac{\chi_n - c^2 t^2}{1-t^2}} dt. \quad (203)$$

We combine (203) with (198) in the proof of Theorem 17 above to obtain

$$\begin{aligned} 1 &< \frac{2}{\pi} \sqrt{\frac{\chi_n - c^2}{1+t_n} + c^2(1-t_n)} \cdot \int_{t_n}^1 \frac{dt}{\sqrt{1-t}} \\ &< \frac{4}{\pi} \sqrt{c^2(1-t_n)^2 + (\chi_n - c^2) \cdot (1-t_n)}. \end{aligned} \quad (204)$$

We rewrite (204) as

$$c^2(1 - t_n)^2 + (\chi_n - c^2) \cdot (1 - t_n) - \frac{\pi^2}{16} > 0. \quad (205)$$

Since  $1 - t_n$  is positive, (205) implies that  $1 - t_n$  is bounded from below by the maximal root  $x_{\max}$  of the quadratic equation

$$c^2 \cdot x^2 + \frac{\chi_n - c^2}{2} \cdot x - \frac{\pi^2}{16} = 0, \quad (206)$$

given via the formula

$$\begin{aligned} x_{\max} &= \frac{1}{2c^2} \cdot \left( \sqrt{(\chi_n - c^2)^2 + \pi^2 c^2 / 4} - (\chi_n - c^2) \right) \\ &= \frac{\pi^2 c^2}{8c^2} \cdot \frac{1}{\chi_n - c^2 + \sqrt{(\chi_n - c^2)^2 + \pi^2 c^2 / 4}}, \end{aligned} \quad (207)$$

which implies (202). ■

## 4.2 Growth Properties of PSWFs

In this subsection, we establish several bounds on  $|\psi_n|$  and  $|\psi'_n|$ . Throughout this subsection  $c > 0$  is a fixed positive real number. The principal results of this subsection are Theorems 19, 20, 21. The following lemma is a technical tool to be used in the rest of this subsection.

**Lemma 14.** *Suppose that  $n \geq 0$  is a non-negative integer, and that the functions  $p, q : \mathbb{R} \rightarrow \mathbb{R}$  are defined via (39) in Section 2.4. Suppose also that the functions  $Q, \tilde{Q} : (0, \min \{ \sqrt{\chi_n}/c, 1 \}) \rightarrow \mathbb{R}$  are defined, respectively, via the formulae*

$$Q(t) = \psi_n^2(t) + \frac{p(t)}{q(t)} \cdot (\psi'_n(t))^2 = \psi_n^2(t) + \frac{(1 - t^2) \cdot (\psi'_n(t))^2}{\chi_n - c^2 t^2} \quad (208)$$

and

$$\begin{aligned} \tilde{Q}(t) &= p(t) \cdot q(t) \cdot Q(t) \\ &= (1 - t^2) \cdot \left( (\chi_n - c^2 t^2) \cdot \psi_n^2(t) + (1 - t^2) \cdot (\psi'_n(t))^2 \right). \end{aligned} \quad (209)$$

Then,  $Q$  is increasing in the interval  $(0, \min \{ \sqrt{\chi_n}/c, 1 \})$ , and  $\tilde{Q}$  is decreasing in the interval  $(0, \min \{ \sqrt{\chi_n}/c, 1 \})$ .

*Proof.* We differentiate  $Q$ , defined via (208), with respect to  $t$  to obtain

$$\begin{aligned} Q'(t) &= 2 \cdot \psi_n(t) \cdot \psi'_n(t) + \left( \frac{2c^2 t \cdot (1 - t^2)}{(\chi_n - c^2 t^2)^2} - \frac{2t}{\chi_n - c^2 t^2} \right) \cdot (\psi'_n(t))^2 + \\ &\quad \frac{2 \cdot (1 - t^2)}{\chi_n - c^2 t^2} \cdot \psi'_n(t) \cdot \psi''_n(t). \end{aligned} \quad (210)$$

Due to (14) in Section 2.1,

$$\psi_n''(t) = \frac{2t}{1-t^2} \cdot \psi_n'(t) - \frac{\chi_n - c^2 t^2}{1-t^2} \cdot \psi_n(t), \quad (211)$$

for all  $-1 < t < 1$ . We substitute (211) into (210) and carry out straightforward algebraic manipulations to obtain

$$Q'(t) = \frac{2t}{(\chi_n - c^2 t^2)^2} \cdot (\chi_n + c^2 - 2c^2 t^2) \cdot (\psi_n'(t))^2. \quad (212)$$

Obviously, for all  $0 < t < \min\{\sqrt{\chi_n}/c, 1\}$ ,

$$\chi_n + c^2 - 2c^2 t^2 > 0. \quad (213)$$

We combine (212) with (213) to conclude that

$$Q'(t) > 0, \quad (214)$$

for all  $0 < t < \min\{\sqrt{\chi_n}/c, 1\}$ . Then, we differentiate  $\tilde{Q}$ , defined via (209), with respect to  $t$  to obtain

$$\begin{aligned} \tilde{Q}'(t) = & -2t \cdot \left( (\chi_n - c^2 t^2) \cdot \psi_n^2(t) + (1 - t^2) \cdot (\psi_n'(t))^2 \right) \\ & + (1 - t^2) \cdot (-2c^2 t \cdot \psi_n^2(t) + 2 \cdot (\chi_n - c^2 t^2) \cdot \psi_n(t) \cdot \psi_n'(t) \\ & - 2t \cdot (\psi_n'(t))^2 + 2 \cdot (1 - t^2) \cdot \psi_n'(t) \cdot \psi_n''(t) \right). \end{aligned} \quad (215)$$

We substitute (211) into (215) and carry out straightforward algebraic manipulations to obtain

$$\tilde{Q}'(t) = -2t \cdot (\chi_n + c^2 - 2c^2 t^2) \cdot \psi_n^2(t). \quad (216)$$

We combine (213) with (216) to conclude that

$$\tilde{Q}'(t) < 0, \quad (217)$$

for all  $0 < t < \min\{\sqrt{\chi_n}/c, 1\}$ . We combine (214) and (217) to finish the proof.  $\blacksquare$

In the following theorem, we establish a lower bound on  $|\psi_n(1)|$ .

**Theorem 19** (bound on  $|\psi_n(1)|$ ). *Suppose that  $\chi_n > c^2$ . Then,*

$$|\psi_n(1)| > \frac{1}{\sqrt{2}}. \quad (218)$$

*Proof.* The function  $Q(t)$  defined by (208) is increasing in  $(0, 1)$  by Lemma 14 and continuous up to  $t = 1$  by Theorem 3 in Section 2.1. Therefore

$$\psi_n^2(t) < Q(t) \leq Q(1) = \psi_n^2(1), \quad 0 \leq t < 1. \quad (219)$$

By Theorem 1 in Section 2.1,

$$\frac{1}{2} = \int_0^1 \psi_n^2(t) dt < \int_0^1 \psi_n^2(1) dt = \psi_n^2(1), \quad (220)$$

which implies (218).  $\blacksquare$

The following theorem describes some of the properties of the extrema of  $\psi_n$  in  $(-1, 1)$ .

**Theorem 20.** *Suppose that  $n \geq 0$  is a non-negative integer, and that  $x, y$  are two arbitrary extremum points of  $\psi_n$  in  $(-1, 1)$ . If  $|x| < |y|$ , then*

$$|\psi_n(x)| < |\psi_n(y)|. \quad (221)$$

*If, in addition,  $\chi_n > c^2$ , then*

$$|\psi_n(x)| < |\psi_n(y)| < |\psi_n(1)|. \quad (222)$$

*Proof.* We observe that  $|\psi_n|$  is even in  $(-1, 1)$ , and combine this observation with the fact that the function  $Q : [-1, 1] \rightarrow \mathbb{R}$ , defined via (208), is increasing in  $(0, 1)$  due to Lemma 14. ■

In the following theorem, we provide an upper bound on the reciprocal of  $|\psi_n|$  (if  $n$  is even) or  $|\psi'_n|$  (if  $n$  is odd) at zero.

**Theorem 21.** *Suppose that  $\chi_n > c^2$ . If  $n$  is even, then*

$$\frac{1}{|\psi_n(0)|} \leq 4 \cdot \sqrt{n \cdot \frac{\chi_n}{c^2}}. \quad (223)$$

*If  $n$  is odd, then*

$$\frac{1}{|\psi'_n(0)|} \leq 4 \cdot \sqrt{\frac{n}{c^2}}. \quad (224)$$

*Proof.* Since  $\chi_n > c^2$ , the inequality

$$\psi_n^2(t) \leq \psi_n^2(1) \leq n + \frac{1}{2}, \quad (225)$$

holds due to Theorem 4 in Section 2.1 and Theorem 20 above. Therefore,

$$\int_{1-1/8n}^1 \psi_n^2(t) dt \leq \frac{1}{8} + \frac{1}{16n} < \frac{3}{16}. \quad (226)$$

Combined with the orthonormality of  $\psi_n$ , this yields the inequality

$$\int_0^{1-1/8n} \psi_n^2(t) dt = \int_0^1 \psi_n^2(t) dt - \int_{1-1/8n}^1 \psi_n^2(t) dt \geq \frac{1}{2} - \frac{3}{16} = \frac{5}{16}. \quad (227)$$

Since

$$\int \frac{dx}{(1-x^2)^2} = \frac{1}{2} \cdot \frac{x}{1-x^2} + \frac{1}{4} \log \frac{x+1}{1-x}, \quad (228)$$

it follows that

$$\begin{aligned}
& \int_0^{1-1/8n} \frac{dx}{(1-x^2)^2} = \\
& \frac{1}{2} \cdot \frac{1-1/8n}{1-(1-1/8n)^2} + \frac{1}{4} \log \frac{2-1/8n}{1/8n} = \\
& \frac{1}{2} \cdot \frac{8n(8n-1)}{16n-1} + \frac{1}{4} \log(16n-1) \leq \\
& 4n + n \leq 5n.
\end{aligned} \tag{229}$$

Suppose that the functions  $Q(t), \tilde{Q}(t)$  are defined for  $-1 \leq t \leq 1$ , respectively, via the formulae (208), (209) in Lemma 14 in Section 4.2. We apply Lemma 14 with  $t_0 = 0$  and  $0 < t \leq 1$  to obtain

$$\begin{aligned}
Q(0) \cdot \chi_n &= Q(0) \cdot p(0) \cdot q(0) = \tilde{Q}(0) \\
&\geq \tilde{Q}(t) = c^2 \left[ \psi_n^2(t) + \frac{(t^2-1)(\psi_n'(t))^2}{(c^2 \cdot t^2 - \chi_n)} \right] \cdot (1-t^2)(\chi_n/c^2 - t^2) \\
&\geq c^2 \psi_n^2(t) (1-t^2)(\chi_n/c^2 - t^2) \geq c^2 \psi_n^2(t) (1-t^2)^2.
\end{aligned} \tag{230}$$

It follows from (227), (229) and (230) that

$$5n \cdot Q(0) \cdot \frac{\chi_n}{c^2} \geq Q(0) \cdot \frac{\chi_n}{c^2} \int_0^{1-1/8n} \frac{dx}{(1-x^2)^2} \geq \int_0^{1-1/8n} \psi_n^2(t) dt \geq \frac{5}{16}, \tag{231}$$

which, in turn, implies that

$$\frac{1}{Q(0)} \leq 16n \cdot \frac{\chi_n}{c^2}. \tag{232}$$

If  $n$  is even, then  $\psi_n'(0) = 0$ , also, if  $n$  is odd, then  $\psi_n(0) = 0$ . Combined with (232), this observation yields both (223) and (224).  $\blacksquare$

## 5 Numerical Results

In this section, we illustrate the analysis of Section 4 via several numerical experiments. All the calculations were implemented in FORTRAN (the Lahey 95 LINUX version) and were carried out in double precision. The algorithms for the evaluation of PSWFs and their eigenvalues were based on [4].

We illustrate Lemma 1 in Figures 1, 2, via plotting  $\psi_n$  with  $\chi_n < c^2$  and  $\chi_n > c^2$ , respectively. The relations (63) and (64) hold for the functions in Figures 1, 2, respectively. Theorem 20 holds in both cases, that is, the absolute value of local extrema of  $\psi_n(t)$  increases as  $t$  grows from 0 to 1. On the other hand, (222) holds only for the function plotted in Figure 2, as expected.

Tables 1, 2, 3 illustrate Theorems 8, 9 in the case  $\chi_n > c^2$ . The band limit  $c > 0$  is fixed per table and chosen to be equal to 10, 100 and 1000, respectively. The first two columns

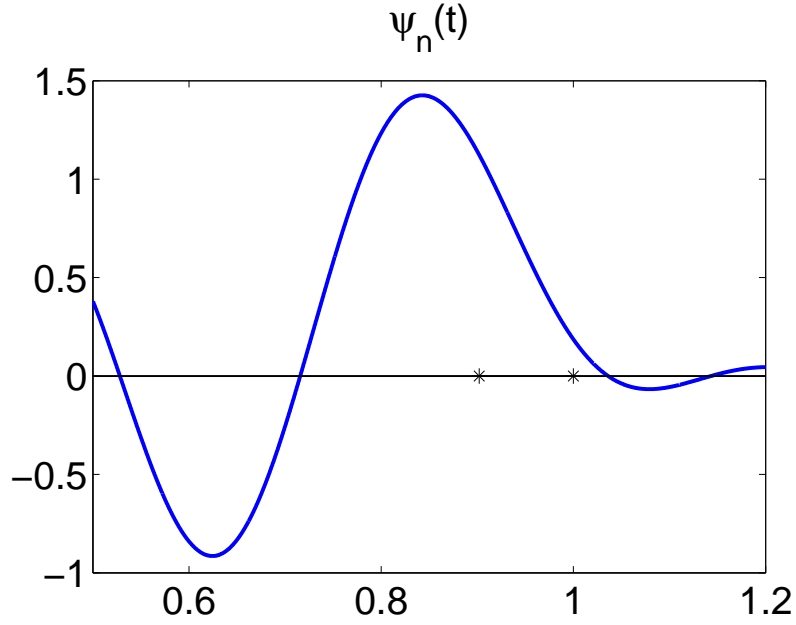


Figure 1: The function  $\psi_n(t)$  for  $c = 20$  and  $n = 9$ . Since  $\chi_n \approx 325.42 < c^2$ , the location of the special points is according to (63) of Lemma 1. The points  $\sqrt{\chi_n}/c \approx 0.90197$  and 1 are marked with asterisks. Compare to Figure 2.

$n$	$\chi_n/c^2$	Above( $n$ )	Below( $n$ )	$\frac{\text{Above}(n)-n}{n}$	$\frac{n-\text{Below}(n)}{n}$
6	0.10104E+01	0.65036E+01	0.59568E+01	0.83927E-01	0.71987E-02
10	0.16310E+01	0.10498E+02	0.99600E+01	0.49826E-01	0.39974E-02
15	0.29137E+01	0.15494E+02	0.14963E+02	0.32940E-01	0.24599E-02
20	0.47078E+01	0.20495E+02	0.19964E+02	0.24737E-01	0.17952E-02
25	0.70050E+01	0.25496E+02	0.24965E+02	0.19820E-01	0.14066E-02
30	0.98035E+01	0.30496E+02	0.29965E+02	0.16538E-01	0.11533E-02
35	0.13103E+02	0.35497E+02	0.34966E+02	0.14189E-01	0.97596E-03
40	0.16902E+02	0.40497E+02	0.39966E+02	0.12425E-01	0.84521E-03
45	0.21202E+02	0.45497E+02	0.44966E+02	0.11052E-01	0.74500E-03

Table 1: Illustration of Theorems 8, 9 with  $c = 10$ . The quantities Above( $n$ ) and Below( $n$ ) are defined by (233).

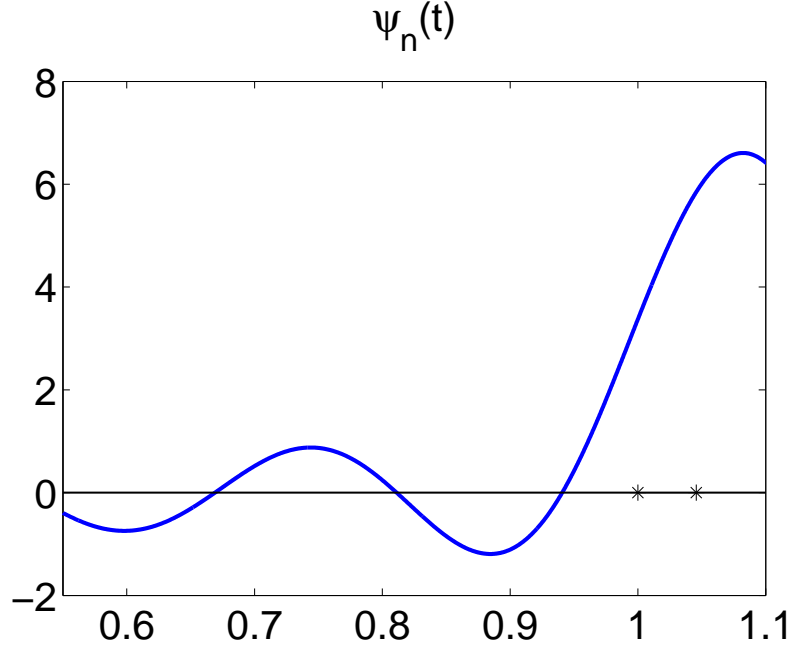


Figure 2: The function  $\psi_n(t)$  for  $c = 20$  and  $n = 14$ . Since  $\chi_n \approx 437.36 > c^2$ , the location of the special points is according to (64) of Lemma 1. The points 1 and  $\sqrt{\chi_n}/c \approx 1.0457$  are marked with asterisks. Compare to Figure 1.

$n$	$\chi_n/c^2$	Above( $n$ )	Below( $n$ )	$\frac{\text{Above}(n)-n}{n}$	$\frac{n-\text{Below}(n)}{n}$
64	0.10066E+01	0.64590E+02	0.63964E+02	0.92169E-02	0.56216E-03
70	0.10668E+01	0.70513E+02	0.69971E+02	0.73216E-02	0.40732E-03
75	0.11290E+01	0.75505E+02	0.74971E+02	0.67341E-02	0.38256E-03
80	0.11989E+01	0.80502E+02	0.79970E+02	0.62812E-02	0.37011E-03
85	0.12756E+01	0.85501E+02	0.84970E+02	0.58974E-02	0.35594E-03
90	0.13584E+01	0.90501E+02	0.89969E+02	0.55623E-02	0.34087E-03
95	0.14472E+01	0.95500E+02	0.94969E+02	0.52652E-02	0.32589E-03
100	0.15416E+01	0.10050E+03	0.99969E+02	0.49994E-02	0.31150E-03

Table 2: Illustration of Theorems 8, 9 with  $c = 100$ . The quantities Above( $n$ ) and Below( $n$ ) are defined by (233).



$n$	$\chi_n/c^2$	Above( $n$ )	Below( $n$ )	$\frac{\text{Above}(n)-n}{n}$	$\frac{n-\text{Below}(n)}{n}$
637	0.10005E+01	0.63759E+03	0.63697E+03	0.93059E-03	0.51797E-04
640	0.10025E+01	0.64055E+03	0.63997E+03	0.85557E-03	0.49251E-04
645	0.10063E+01	0.64552E+03	0.64497E+03	0.80101E-03	0.39996E-04
650	0.10105E+01	0.65051E+03	0.64997E+03	0.78412E-03	0.39578E-04
655	0.10149E+01	0.65551E+03	0.65497E+03	0.77352E-03	0.40527E-04
660	0.10195E+01	0.66050E+03	0.65997E+03	0.76512E-03	0.41359E-04
665	0.10243E+01	0.66550E+03	0.66497E+03	0.75777E-03	0.41942E-04
670	0.10292E+01	0.67050E+03	0.66997E+03	0.75103E-03	0.42321E-04
675	0.10343E+01	0.67550E+03	0.67497E+03	0.74469E-03	0.42547E-04

Table 3: *Illustration of Theorems 8, 9*  $c = 1000$ . The quantities Above( $n$ ) and Below( $n$ ) are defined by (233).

contain  $n$  and the ratio  $\chi_n/c^2$ . The third and fourth column contain the upper and lower bound on  $n$  given, respectively, via (99) in Theorem 8 and (150) in Theorem 9, i.e.

$$\begin{aligned}
\text{Below}(n) &= 1 + \frac{2}{\pi} \int_0^{t_n} \sqrt{\frac{\chi_n - c^2 t^2}{1 - t^2}} dt = 1 + \frac{2}{\pi} \sqrt{\chi_n} \cdot E\left(\text{asin}(t_n), \frac{c}{\sqrt{\chi_n}}\right) \\
\text{Above}(n) &= \frac{2}{\pi} \int_0^1 \sqrt{\frac{\chi_n - c^2 t^2}{1 - t^2}} dt = \frac{2}{\pi} \sqrt{\chi_n} \cdot E\left(\frac{c}{\sqrt{\chi_n}}\right),
\end{aligned} \tag{233}$$

where  $E$  denote the elliptical integrals of Section 2.2, and  $t_n$  is the maximal root of  $\psi_n$  in  $(-1, 1)$  (see also (154)). The fifth and sixth columns contain the relative errors of these bounds. The first row corresponds to the minimal  $n$  for which  $\chi_n > c^2$ . We observe that for a fixed  $c$  the bounds become more accurate as  $n$  grows. Also, for  $n = \lceil 2c/\pi \rceil + 1$  the accuracy improves as  $c$  grows. Moreover, the lower bound is always more accurate than the upper bound.

$n$	$\chi_n/c^2$	Above( $n$ )	Below( $n$ )	$\frac{\text{Above}(n)-n}{n}$	$\frac{n-\text{Below}(n)}{n}$
1	0.29824E-01	0.10395E+01	0.10000E+01	0.39511E-01	0.00000E+00
9	0.18531E+00	0.90625E+01	0.89818E+01	0.69444E-02	0.20214E-02
19	0.36985E+00	0.19069E+02	0.18981E+02	0.36421E-02	0.10180E-02
29	0.54240E+00	0.29075E+02	0.28980E+02	0.25825E-02	0.69027E-03
39	0.70125E+00	0.39082E+02	0.38979E+02	0.21102E-02	0.53327E-03
49	0.84356E+00	0.49096E+02	0.48978E+02	0.19543E-02	0.45122E-03
54	0.90685E+00	0.54110E+02	0.53977E+02	0.20330E-02	0.43263E-03
59	0.96278E+00	0.59146E+02	0.58974E+02	0.24725E-02	0.44189E-03
63	0.99867E+00	0.63420E+02	0.62966E+02	0.66661E-02	0.53355E-03

Table 4: *Illustration of Theorems 8, 9* with  $c = 100$ . The quantities Above( $n$ ) and Below( $n$ ) are defined by (234).

Table 4 illustrates Theorems 8, 9 in the case  $\chi_n < c^2$  with  $c = 100$ . The structure of Table 4 is the same as that of Tables 1, 2, 3 with the only difference: the third and fourth column contain the upper and lower bound on  $n$  given, respectively, via (100) in Theorems 8 and (150) in Theorem 9, i.e.

$$\begin{aligned} \text{Below}(n) &= 1 + \frac{2}{\pi} \int_0^{t_n} \sqrt{\frac{\chi_n - c^2 t^2}{1 - t^2}} dt = 1 + \frac{2}{\pi} \sqrt{\chi_n} \cdot E\left(\text{asin}(t_n), \frac{c}{\sqrt{\chi_n}}\right) \\ \text{Above}(n) &= \frac{2}{\pi} \int_0^T \sqrt{\frac{\chi_n - c^2 t^2}{1 - t^2}} dt = \frac{2}{\pi} \sqrt{\chi_n} \cdot E\left(\text{asin}(T), \frac{c}{\sqrt{\chi_n}}\right), \end{aligned} \quad (234)$$

where  $t_n$  and  $T$  are the maximal roots of  $\psi_n$  and  $\psi'_n$  in the interval  $(-1, 1)$ , respectively. The values in the first row grow up to  $\lfloor 2c/\pi \rfloor$ , in correspondence with Theorem 14 in Section 4.1.2. We observe that both bounds in the third and fourth columns are correct and the lower bound is always more accurate. This behavior is similar to that observed in Tables 1, 2, 3.

$n$	$(n - 2c/\pi - 1)/c$	$\chi_n$	$(\frac{\pi}{2}(n+1))^2$	$(\frac{\pi}{2}(n+1))^2/\chi_n - 1$
640	0.23802E-02	0.10025E+07	0.10138E+07	0.11248E-01
650	0.12380E-01	0.10105E+07	0.10457E+07	0.34836E-01
660	0.22380E-01	0.10195E+07	0.10781E+07	0.57443E-01
670	0.32380E-01	0.10292E+07	0.11109E+07	0.79396E-01
680	0.42380E-01	0.10395E+07	0.11443E+07	0.10082E+00
690	0.52380E-01	0.10503E+07	0.11781E+07	0.12177E+00
700	0.62380E-01	0.10615E+07	0.12125E+07	0.14229E+00
710	0.72380E-01	0.10731E+07	0.12473E+07	0.16241E+00
720	0.82380E-01	0.10850E+07	0.12827E+07	0.18215E+00
730	0.92380E-01	0.10973E+07	0.13185E+07	0.20152E+00
740	0.10238E+00	0.11100E+07	0.13548E+07	0.22054E+00
750	0.11238E+00	0.11230E+07	0.13916E+07	0.23922E+00
760	0.12238E+00	0.11363E+07	0.14289E+07	0.25757E+00
770	0.13238E+00	0.11498E+07	0.14667E+07	0.27559E+00
780	0.14238E+00	0.11637E+07	0.15050E+07	0.29330E+00
790	0.15238E+00	0.11779E+07	0.15438E+07	0.31069E+00
800	0.16238E+00	0.11923E+07	0.15831E+07	0.32777E+00
810	0.17238E+00	0.12070E+07	0.16229E+07	0.34456E+00
820	0.18238E+00	0.12219E+07	0.16631E+07	0.36105E+00

Table 5: *Illustration of Theorem 12 with  $c = 1000$ .*

Tables 5, 6 illustrate Theorem 12 with  $c = 1000$  and  $c = 10000$ , respectively. The first column contains the PSWF index  $n$ , which starts from roughly  $2c/\pi$  and increases by steps of  $c/10$ . The second column displays the normalized distance  $d_n$  between  $n$  and  $(2c/\pi + 1)$ , defined via the formula

$$d_n = \frac{n - 2c/\pi - 1}{c}. \quad (235)$$

$n$	$(n - 2c/\pi - 1)/c$	$\chi_n$	$(\frac{\pi}{2}(n+1))^2$	$(\frac{\pi}{2}(n+1))^2/\chi_n - 1$
6400	0.32802E-02	0.10022E+09	0.10110E+09	0.87670E-02
6500	0.13280E-01	0.10101E+09	0.10428E+09	0.32378E-01
6600	0.23280E-01	0.10191E+09	0.10751E+09	0.55007E-01
6700	0.33280E-01	0.10288E+09	0.11079E+09	0.76977E-01
6800	0.43280E-01	0.10390E+09	0.11413E+09	0.98410E-01
6900	0.53280E-01	0.10498E+09	0.11751E+09	0.11937E+00
7000	0.63280E-01	0.10609E+09	0.12094E+09	0.13991E+00
7100	0.73280E-01	0.10725E+09	0.12442E+09	0.16004E+00
7200	0.83280E-01	0.10845E+09	0.12795E+09	0.17979E+00
7300	0.93280E-01	0.10968E+09	0.13152E+09	0.19918E+00
7400	0.10328E+00	0.11094E+09	0.13515E+09	0.21821E+00
7500	0.11328E+00	0.11224E+09	0.13883E+09	0.23691E+00
7600	0.12328E+00	0.11357E+09	0.14255E+09	0.25526E+00
7700	0.13328E+00	0.11492E+09	0.14633E+09	0.27330E+00
7800	0.14328E+00	0.11631E+09	0.15016E+09	0.29102E+00
7900	0.15328E+00	0.11772E+09	0.15403E+09	0.30842E+00
8000	0.16328E+00	0.11916E+09	0.15795E+09	0.32552E+00
8100	0.17328E+00	0.12063E+09	0.16193E+09	0.34232E+00
8200	0.18328E+00	0.12213E+09	0.16595E+09	0.35883E+00

Table 6: *Illustration of Theorem 12 with  $c = 10000$ .*

The third column contains  $\chi_n$ . The fourth and fifth column contain the upper bound on  $\chi_n$ , defined in Theorem 12, and the relative error of this bound, respectively. We observe that the bound is slightly better for  $c = 10000$ , if we keep  $d_n$  fixed, and deteriorates as  $n$  grows for a fixed  $c$ . In fact, starting from  $n \approx (2/\pi + 1/6) \cdot c$ , this bound becomes even worse than (15) (this value is  $n = 825$  for  $c = 1000$  and  $n = 8254$  for  $c = 10000$ ). Since Theorem 12 is a simplification of more accurate Theorems 8, 9, the latter observation is not surprising. Nevertheless, the high accuracy for  $n \approx 2c/\pi$  and the simplicity of the estimate make Theorem 12 useful.

$i$	$t_{i+1} - t_i$	$\frac{\pi}{f(t_{i+1})+v(t_{i+1})/2}$	$\frac{\pi}{f(t_i)}$	lower error	upper error
44	0.27468E-01	0.27464E-01	0.27470E-01	0.13152E-03	0.63357E-04
45	0.27463E-01	0.27454E-01	0.27467E-01	0.32708E-03	0.15253E-03
46	0.27453E-01	0.27439E-01	0.27460E-01	0.52432E-03	0.24265E-03
47	0.27438E-01	0.27418E-01	0.27447E-01	0.72429E-03	0.33437E-03
60	0.26685E-01	0.26573E-01	0.26741E-01	0.42160E-02	0.21008E-02
61	0.26568E-01	0.26444E-01	0.26630E-01	0.46314E-02	0.23349E-02
62	0.26437E-01	0.26303E-01	0.26506E-01	0.50867E-02	0.25968E-02
63	0.26293E-01	0.26146E-01	0.26369E-01	0.55889E-02	0.28916E-02
70	0.24700E-01	0.24418E-01	0.24863E-01	0.11404E-01	0.66360E-02
71	0.24347E-01	0.24036E-01	0.24533E-01	0.12811E-01	0.76073E-02
72	0.23948E-01	0.23602E-01	0.24158E-01	0.14473E-01	0.87772E-02
73	0.23493E-01	0.23107E-01	0.23733E-01	0.16457E-01	0.10201E-01
83	0.12096E-01	0.10707E-01	0.13206E-01	0.11484E+00	0.91691E-01
84	0.96757E-02	0.81279E-02	0.10948E-01	0.15996E+00	0.13147E+00
85	0.69453E-02	0.52650E-02	0.83714E-02	0.24194E+00	0.20533E+00
86	0.39568E-02	0.22125E-02	0.55074E-02	0.44083E+00	0.39188E+00

Table 7: *Illustration of Theorem 10 with  $c = 100$  and  $n = 87$ .*

Tables 7, 8 illustrate Theorems 10, 15, with  $c = 100, n = 87$  and  $c = 1000, n = 670$ , respectively. The first column contains the index  $i$  of the  $i$ th root  $t_i$  of  $\psi_n$  inside  $(-1, 1)$ . The second column contains the difference between two consecutive roots  $t_{i+1}$  and  $t_i$ . The third and fourth columns contain, respectively, the lower and upper bounds on this difference, given via (155) in Theorem 10. The last two columns contain the relative errors of these bounds. We observe that both estimates are fairly accurate when  $t_i$  is far from 1, and the accuracy increases with  $c$ . The best relative accuracy is about 0.01% for  $c = 100$  and 0.0001% for  $c = 1000$ . Both bounds deteriorate as  $i$  grows to  $n$ . For both values of  $c$  the relative accuracy of the lower bound for  $i = n - 1$  is as low as 44%, and that of the upper bound is about 39%. In general, the upper bound is always more accurate. We also note that  $t_{i+1} - t_i$  decreases monotonically as  $i$  grows, which confirms Theorem 15, since  $\chi_n > c^2$  in both cases.

$i$	$t_{i+1} - t_i$	$\frac{\pi}{f(t_{i+1})+v(t_{i+1})/2}$	$\frac{\pi}{f(t_i)}$	lower error	upper error
336	0.30967E-02	0.30967E-02	0.30967E-02	0.19367E-05	0.59233E-06
337	0.30967E-02	0.30967E-02	0.30967E-02	0.35775E-05	0.72845E-06
338	0.30967E-02	0.30967E-02	0.30967E-02	0.52185E-05	0.86461E-06
339	0.30967E-02	0.30967E-02	0.30967E-02	0.68599E-05	0.10008E-05
400	0.30948E-02	0.30945E-02	0.30949E-02	0.11172E-03	0.10078E-04
401	0.30948E-02	0.30944E-02	0.30948E-02	0.11359E-03	0.10252E-04
402	0.30947E-02	0.30944E-02	0.30947E-02	0.11547E-03	0.10427E-04
403	0.30947E-02	0.30943E-02	0.30947E-02	0.11735E-03	0.10603E-04
500	0.30813E-02	0.30802E-02	0.30815E-02	0.37302E-03	0.41125E-04
501	0.30811E-02	0.30799E-02	0.30812E-02	0.37699E-03	0.41713E-04
502	0.30808E-02	0.30797E-02	0.30810E-02	0.38101E-03	0.42311E-04
503	0.30806E-02	0.30794E-02	0.30807E-02	0.38507E-03	0.42920E-04
600	0.30127E-02	0.30084E-02	0.30136E-02	0.14255E-02	0.29783E-03
601	0.30109E-02	0.30065E-02	0.30118E-02	0.14549E-02	0.30734E-03
602	0.30090E-02	0.30045E-02	0.30099E-02	0.14853E-02	0.31731E-03
603	0.30071E-02	0.30025E-02	0.30080E-02	0.15168E-02	0.32775E-03
666	0.12704E-02	0.11248E-02	0.13859E-02	0.11465E+00	0.90887E-01
667	0.10176E-02	0.85504E-03	0.11505E-02	0.15973E+00	0.13065E+00
668	0.73133E-03	0.55454E-03	0.88094E-03	0.24173E+00	0.20458E+00
669	0.41703E-03	0.23323E-03	0.58020E-03	0.44073E+00	0.39128E+00

Table 8: *Illustration of Theorem 10 with  $c = 1000$  and  $n = 670$ .*

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